Ferromagnetic Domain Wall and Spiral Ground States in One-Dimensional Deformed Flat-Band Hubbard Model

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We construct a set of exact ground states with a localized ferromagnetic domain wall and an extended spiral structure in a quasi-one-dimensional deformed flat-band Hubbard model. In the case of quarter filling, we show the uniqueness of the ground state with a fixed magnetization. The ground states with these structures are degenerate with the all-spin-up and all-spin-down states. This property of the degeneracy is the same as the domain wall solutions in the XXZ Heisenberg–Ising model. We derive a useful recursion relation for the normalization of the domain wall ground state. Using this recursion relation, we discuss the convergence of the ground state expectation values of arbitrary local operators in the infinite-volume limit. In the ground state of the infinite-volume system, the translational symmetry is spontaneously broken by this structure. We prove that the cluster property holds for the domain wall ground state and excited states. We also estimate bounds of the ground state expectation values of several observables, such as one- and two-point functions of spin and electron number density.

KEY WORDS: Ferromagnetic domain wall; spiral state; flat-band Hubbard model; exact solution; quantum effect; cluster property.

1. INTRODUCTION

Appearance of domain structure is known to be a universal phenomenon in ferromagnetic systems with an energy gap and with a finite correlation length. In realistic materials, the main driving force to produce the ferromagnetic domains is the magnetic dipole–dipole interaction, which is much

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smaller than the Coulomb interaction but long-ranged. In some examples we introduce here, however, stable domain structures appear without such long range interactions. Alcaraz, Salinas and Wreszinski construct a set of ground states with a domain structure in the ferromagnetic XXZ model with a critical boundary field in arbitrary dimensions for an arbitrary spin.[1] They have shown that the degeneracy of the ground states corresponding to the location of a domain wall center is identical to that of the ground states in the SU(2) invariant model. In their solution, the domain wall is localized at an arbitrary location with a finite width depending on the Ising anisotropy parameter $(q + q^{-1})/2 > 1$ with a positive number q. After their construction of the domain wall ground state, lots of interesting properties of the solution have been discovered. Here, we focus only on one-dimensional models and remark several interesting results. Koma and Nachtergaele have proved the existence of the energy gap which is independent of the location of the domain wall[2] in the one-dimensional XXZ model with a spin $S = 1/2$. Matsui has obtained the complete set of pure ground states in this model in the infinite-volume limit.[3] Koma and Nachtergaele have also obtained the complete set of pure ground states with domain walls in the one-dimensional XXZ model with an arbitrary spin S in the infinite-volume limit.[4] Koma, Nachtergaele and Starr have shown a singular excitation localized near the domain wall with lower energy. Bach and Macris evaluate a spin one-point function in the domain wall state of the one-dimensional XXZ model by a rigorous perturbation method.[5] Datta and Kennedy discuss the stability of a domain wall in one-dimensional quantum spin models at zero temperature by another rigorous perturbation method Kirkwood–Thomas approach around the Ising model as an unperturbed model.[6] In the antiferromagnetic XXZ model, the exchange interaction destabilizes the domain structure, while in the ferromagnetic model it preserves the domain structure at zero temperature. The role of the quantum effects should be studied more in many other models.

In many-electron systems, stable ferromagnetism itself is less trivial than that in spin models. The ferromagnetism appears by purely quantum effects in many-electron systems. There are no corresponding classical systems to the ferromagnetic many-electron systems. Some remarkable results for ferromagnetic ground states have recently been obtained in a class of Hubbard models. Mielke has shown that the ground states give saturated ferromagnetism in many-electron models on general line graphs with special properties.[7] Independently, Tasaki has shown that ground states give the saturated ferromagnetism in another class of models.[8] These models are called the flat-band Hubbard models, since the band structure for a single electron in the carefully constructed hopping Hamiltonian becomes

flat. Mielke and Tasaki argue the sufficient conditions for the flat-band ferromagnetism in extended models.[9] Tasaki has proved that the saturated ferromagnetism is stable against a perturbation which bends the electron band.[10] Tanaka and Ueda have shown the stability of the saturated ferromagnetism in a more complicated two-dimensional model in Mielke's class.[11] The ferromagnetism in flat-band Hubbard models and their perturbed models is believed to be one universal nature of ferromagnetism in many-electron systems. In these models, the ferromagnetism appears in non-perturbative region in which the Coulomb repulsion between electrons is sufficiently strong. No perturbative expansion method can explain the ferromagnetism in these models. Only non-perturbative methods particularly devised can work for these problems.

In this paper, we construct a set of exact ground states in a one-dimensional flat-band Hubbard model with coexisting domain wall and spiral structures. We deform a flat-band Hubbard model by introducing a complex anisotropy parameter q. The SU(2) spin rotation symmetry in the original flat-band model is reduced to U(1) symmetry in our deformed model. This anisotropy $|q| \neq 1$ leads to a localized domain wall with a finite width and a complex q leads to an extended spiral state. We see that the domain wall ground state in our electron model has the same degeneracy as that in the XXZ model and the same localization property in a certain parameter regime. As in the XXZ model, the domain structure is characterized in terms of the local order parameter $\langle S_x^{(3)} \rangle$, which represents the third component of the localized spin at site x. This local order parameter takes the same sign within one domain. The domain wall center is a set of sites x defined by zeros of the local order parameter $\langle S_x^{(3)} \rangle = 0$. For complex $q = |q|e^{i\theta}$ with real θ, our electron model differs from the XXZ quantum spin model which has no spiral ground state. The ground state in our model has an incommensurate spiral structure with a pitch angle θ. In the case of quarter filling, we show the uniqueness of the ground state with a fixed magnetization for an arbitrary q.

To calculate the ground state expectation values of observables, we introduce a normalization function of the ground state. In particular in the one-dimensional model, we derive a recursion relation for the normalization function, which determines this function together with its boundary data. This recursion relation is a linear equation in the unknown function of a site with a function coefficient which depends on the distance from the domain wall center. This method is similar to the method for the normalization function in the non-product domain wall ground states in the XXZ model obtained by Bolina, Contucci and Nachtergaele.[12,13] We treat this recursion relation by a kind of perturbative

expansion method in the function coefficient. This treatment of the recursion relation enables us to discuss the convergence of the ground state expectation values of local observables in the infinite-volume limit. Then, we prove that the cluster property holds for the domain wall ground state and excited states. This property implies that the obtained representation is healthy as a quantum system with infinite degrees of freedom. As discussed in,[14] we expect that quantum effects in the Hubbard model stabilize the domain wall as in the XXZ model. Finally, we show exponential decay of the truncated correlation functions, and evaluate their decay length.

This paper is organized as follows. In Section 2, we define a deformed flat-band model on a quasi-one-dimensional lattice. We give an explicit form of a Hamiltonian. In Section 3, we construct ground states, and prove the uniqueness of the ground state with a fixed magnetization. We also introduce different basis of ground states whose state is superposed over all magnetizations with certain weights. In Section 4, we define a normalization function and extract its convergent factor, which gives us good representations of the expectation values of arbitrary local operators in the ground state. We obtain a recursion relation for the normalization function which is useful to prove important properties of the observables. In Section 5, we evaluate the expectation values of local operators in the infinite-volume limit. We prove the cluster property of the domain wall ground state on the basis of the recursion relation.

2. THE MODEL

2.1. Definition of the Model

We define a one-dimensional lattice Λ with a "length" L by

$$\Lambda := \left\{ x = \frac{y}{2} \,\middle|\, y \in \mathbb{Z} \text{ with } |y| \leqslant L \right\}, \tag{1}$$

where $L = 2l + 1$ is an odd integer (Fig. 1). Λ can be decomposed two sub lattices Λ_o and Λ' defined by

$$\Lambda_o := \left\{ x \in \Lambda \,|\, x \text{ is an integer} \right\} \tag{2}$$

and

$$\Lambda' := \left\{ x \in \Lambda \,|\, x \text{ is a half-odd-integer} \right\}. \tag{3}$$

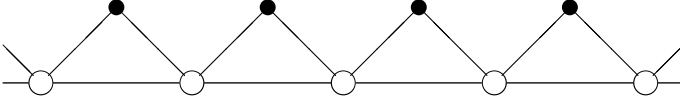Fig. 1.   The lattice Λ. White circles are sites in Λ_o and black dots are sites in Λ'. Electrons at a site can hop to other site if the site is connected to the original site with a line.

   We define a deformed flat-band Hubbard model on the lattice Λ. We denote creation and annihilation operators by $c^{\dagger}_{x,\sigma}$ and $c_{x,\sigma}$, respectively, for an electron at site $x \in \Lambda$ with spin $\sigma = \uparrow, \downarrow$. They satisfy the standard anticommutation relations

$$\{c_{x,\sigma}, c^{\dagger}_{y,\tau}\} = \delta_{x,y}\delta_{\sigma,\tau}, \qquad \{c_{x,\sigma}, c_{y,\tau}\} = \{c^{\dagger}_{x,\sigma}, c^{\dagger}_{y,\tau}\} = 0 \qquad (4)$$

for $x, y \in \Lambda$ and $\sigma, \tau = \uparrow, \downarrow$, where $\{A, B\} = AB + BA$. The electron number operator is defined by

$$n_{x,\sigma} := c^{\dagger}_{x,\sigma}c_{x,\sigma}, \qquad (5)$$

whose eigenvalue gives the number of electron with spin σ at site x. In the present paper, we use the open boundary condition. This condition is given by defining $c_{x,\sigma} = 0$ for $|x| > L/2$. We consider a Hubbard model with a Hamiltonian

$$H = H_{\text{hop}} + H_{\text{int}}. \qquad (6)$$

The interaction Hamiltonian is given by

$$H_{\text{int}} = U \sum_{x \in \Lambda} n_{x,\uparrow}n_{x,\downarrow} \qquad (7)$$

with a positive number U which represents an on-site Coulomb repulsion energy. The hopping Hamiltonian is defined by

$$H_{\text{hop}} = t \sum_{\sigma=\uparrow,\downarrow} \sum_{x \in \Lambda'} \left[(q^{-p(\sigma)/4})^* c^{\dagger}_{x-1/2,\sigma} + \lambda c^{\dagger}_{x,\sigma} + (q^{p(\sigma)/4})^* c^{\dagger}_{x+1/2,\sigma} \right]$$
$$\times \left[q^{-p(\sigma)/4}c_{x-1/2,\sigma} + \lambda c_{x,\sigma} + q^{p(\sigma)/4}c_{x+1/2,\sigma} \right], \quad (8)$$

where $p(\sigma) = 1$ if $\sigma = \uparrow$ and $p(\sigma) = -1$ if $\sigma = \downarrow$ with $t > 0$. We can represent the hopping Hamiltonian also in a standard form with a hopping matrix $t_{x,y}^{(\sigma)}$

$$H_{\mathrm{hop}} = \sum_{\sigma=\uparrow,\downarrow} \sum_{x,y\in\Lambda} t_{x,y}^{(\sigma)} c_{x,\sigma}^{\dagger} c_{y,\sigma}, \tag{9}$$

where

$$\frac{t_{x,y}^{(\sigma)}}{t} = \frac{(t_{y,x}^{(\sigma)})^{*}}{t} = \begin{cases} |q|^{(1/2)} + |q|^{-(1/2)} & \text{if } x = y \in \Lambda_{\mathrm{o}}, \\ \lambda^{2} & \text{if } x = y \in \Lambda', \\ \lambda q^{-p(\sigma)/4} & \text{if } x = y + \frac{1}{2} \text{ and } y \in \Lambda_{\mathrm{o}}, \\ \lambda q^{p(\sigma)/4} & \text{if } x = y - \frac{1}{2} \text{ and } y \in \Lambda_{\mathrm{o}}, \\ (q/q^{*})^{p(\sigma)/4} & \text{if } x = y + 1 \in \Lambda_{\mathrm{o}}, \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

The Hilbert space of the model is spanned by the following basis:

$$\Phi_{\mathcal{A},\mathcal{B}} = \left(\prod_{x\in\mathcal{A}} c_{x,\uparrow}^{\dagger}\right) \left(\prod_{y\in\mathcal{B}} c_{y,\downarrow}^{\dagger}\right) \Phi_{\mathrm{vac}}, \tag{11}$$

where $\mathcal{A}, \mathcal{B}$ are subsets of Λ and Φ_{vac} is the unique normalized state with no electrons defined by $c_{x,\sigma}\Phi_{\mathrm{vac}} = 0$ for any $x \in \Lambda$ and $\sigma = \uparrow, \downarrow$. In the present work, we consider the Hilbert space $\mathcal{H}$ with the electron number fixed to $L = |\Lambda_{\mathrm{o}}|$, which is spanned by the basis states (11) with $|\mathcal{A}| + |\mathcal{B}| = L$.[2]

2.2. Symmetries of the Model Hamiltonian

Let us discuss the symmetries in our model. The local spin operator $S_{x}^{(j)}$ and the total-spin operator $S_{\mathrm{tot}}^{(j)}$ are defined by

$$S_{x}^{(j)} = \frac{1}{2} \sum_{\sigma,\tau} c_{x,\sigma}^{\dagger} (\mathcal{P}^{(j)})_{\sigma,\tau} c_{x,\tau}, \tag{12}$$

and

$$S_{\mathrm{tot}}^{(j)} = \sum_{x\in\Lambda} S_{x}^{(j)} \tag{13}$$

[2]Throughout this paper, $|\mathcal{A}|$ denotes the number of all elements in a set $\mathcal{A}$.

with $j = 1, 2, 3$, where $\mathcal{P}^{(j)}$ is the Pauli matrix

$$\mathcal{P}^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathcal{P}^{(2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \mathcal{P}^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (14)$$

The first important symmetry is U(1) symmetry. The Hamiltonian commutes with $S_{\text{tot}}^{(3)}$, and an eigenvalue of $S_{\text{tot}}^{(3)}$ is called a magnetization which takes $\pm 1/2$, $\pm 3/2$, ..., $\pm L/2$. We define $U(1)$ transformation by a following map U_φ which is generated by $S_{\text{tot}}^{(3)}$ in a finite system:

$$U_\varphi(c_{x,\sigma}^\dagger) = e^{i \frac{\varphi}{2} p(\sigma)} c_{x,\sigma}^\dagger, \qquad U_\varphi(c_{x,\sigma}) = e^{-i \frac{\varphi}{2} p(\sigma)} c_{x,\sigma}, \qquad (15)$$

where φ is a real parameter. This map is linear

$$U_\varphi(A + B) = U_\varphi(A) + U_\varphi(B) \qquad (16)$$

and homomorphic

$$U_\varphi(AB) = U_\varphi(A) U_\varphi(B), \qquad (17)$$

for arbitrary operators A and B. Each spin operator is transformed to the following form under the $U(1)$ transformation:

$$U_\varphi(S_x^{(1)}) = \cos \varphi \ S_x^{(1)} - \sin \varphi \ S_x^{(2)}, \qquad (18)$$
$$U_\varphi(S_x^{(2)}) = \sin \varphi \ S_x^{(1)} + \cos \varphi \ S_x^{(2)}, \qquad (19)$$
$$U_\varphi(S_x^{(3)}) = S_x^3. \qquad (20)$$

Note that the $U(1)$ symmetry is enhanced to the SU(2) in the case of $q = 1$, i.e., the Hamiltonian commutes with the all total-spin operators $S_{\text{tot}}^{(j)}$ ($j = 1, 2, 3$).

The second important symmetry is a $\mathbb{Z}_2$ symmetry defined by a linear and homomorphic map Π

$$\Pi(c_{x,\sigma}) = c_{-x,\overline{\sigma}}, \qquad \Pi(c_{x,\sigma}^\dagger) = c_{-x,\overline{\sigma}}^\dagger, \qquad (21)$$

where $\overline{\sigma}$ takes $\downarrow$ if $\sigma = \uparrow$ and $\uparrow$ if $\sigma = \downarrow$. Note the $\mathbb{Z}_2$ transformation of the total spin operator $\Pi(S_{\text{tot}}^{(3)}) = -S_{\text{tot}}^{(3)}$. An energy eigenstate with a magnetization M is transformed by Π into another eigenstate with a total magnetization $-M$, which belongs to the same energy eigenvalue.

The Hamiltonian of a system with a large volume is invariant under a translation

$$T_u(c_{x,\sigma}) = c_{x+u,\sigma}, \qquad T_u(c_{x,\sigma}^{\dagger}) = c_{x+u,\sigma}^{\dagger}, \tag{22}$$

except several terms near the boundaries, where u is an integer. Thus, the translational symmetry appears after taking the infinite-volume limit.

3. GROUND STATE OF THE MODEL

In this section, we construct ground states of the deformed flat-band Hubbard model. The construction method is based on the original method obtained by Mielke and Tasaki for the positive semi-definite Hamiltonian in the flat-band Hubbard model.[7,8]

3.1. Localized Electron Operators

We define localized electron operators $a_{x,\sigma}^{\dagger}$ and $d_{x,\sigma}$ by

$$a_{x,\sigma}^{\dagger} = \begin{cases} -q^{p(\sigma)/4}c_{x-1/2,\sigma}^{\dagger} + \lambda c_{x,\sigma}^{\dagger} - q^{-p(\sigma)/4}c_{x+1/2,\sigma}^{\dagger} & \text{if } x \in \Lambda_{\mathrm{o}}, \\ \dfrac{1}{\lambda}c_{x,\sigma}^{\dagger} & \text{if } x \in \Lambda', \end{cases} \tag{23}$$

and

$$d_{x,\sigma} = \begin{cases} \dfrac{1}{\lambda}c_{x,\sigma} & \text{if } x \in \Lambda_{\mathrm{o}}, \\ q^{-p(\sigma)/4}c_{x-1/2,\sigma} + \lambda c_{x,\sigma} + q^{p(\sigma)/4}c_{x+1/2,\sigma} & \text{if } x \in \Lambda', \end{cases} \tag{24}$$

$c_{x,\sigma}^{\dagger}$ and $c_{x,\sigma}$ can be written in terms of $a_{x,\sigma}^{\dagger}$ and $d_{x,\sigma}$:

$$c_{x,\sigma}^{\dagger} = \begin{cases} q^{p(\sigma)/4}a_{x-1/2,\sigma}^{\dagger} + \dfrac{1}{\lambda}a_{x,\sigma}^{\dagger} + q^{-p(\sigma)/4}a_{x+1/2,\sigma}^{\dagger} & \text{if } x \in \Lambda_{0}, \\ \lambda a_{x,\sigma}^{\dagger} & \text{if } x \in \Lambda', \end{cases} \tag{25}$$

and

$$c_{x,\sigma} = \begin{cases} \lambda d_{x,\sigma} & \text{if } x \in \Lambda_{\mathrm{o}}, \\ -q^{-p(\sigma)/4}d_{x-1/2,\sigma} + \dfrac{1}{\lambda}d_{x,\sigma} - q^{p(\sigma)/4}d_{x+1/2,\sigma} & \text{if } x \in \Lambda', \end{cases} \tag{26}$$

For an open boundary condition, we define $c_{x,\sigma} = 0$ for $|x| > L/2$ which means $a_{x,\sigma}^\dagger = 0$ and $d_{x,\sigma} = 0$ for $|x| > L/2$. Relations (23)–(26) are also valid consistently under this condition. The inverse relations (25) and (26) implies that the Hilbert space $\mathcal{H}$ can be spanned also by the basis states

$$\Phi'_{\mathcal{A},\mathcal{B}} = \left(\prod_{x \in \mathcal{A}} a_{x,\uparrow}^\dagger \right) \left(\prod_{y \in \mathcal{B}} a_{y,\downarrow}^\dagger \right) \Phi_{\text{vac}}, \tag{27}$$

where $\mathcal{A}$ and $\mathcal{B}$ are subsets of Λ with $|\mathcal{A}| + |\mathcal{B}| = L$. Note that $a_{x,\sigma}^\dagger$ and $d_{x,\sigma}$ satisfy the canonical anticommutation relations

$$\{a_{x,\sigma}^\dagger, d_{y,\tau}\} = \delta_{x,y}\delta_{\sigma,\tau}, \qquad \{a_{x,\sigma}^\dagger, a_{y,\tau}^\dagger\} = \{d_{x,\sigma}, d_{y,\tau}\} = 0. \tag{28}$$

The hopping Hamiltonian can be written in terms of $d_{x,\sigma}$,

$$H_{\text{hop}} = \sum_{\sigma = \uparrow, \downarrow} \sum_{x \in \Lambda'} d_{x,\sigma}^\dagger d_{x,\sigma}. \tag{29}$$

This form of the hopping Hamiltonian indicates the positive semi-definiteness $H_{\text{hop}} \geqslant 0$. The positive semi-definiteness of the interaction Hamiltonian $H_{\text{int}} \geqslant 0$ is also clear because $n_{x,\sigma} = c_{x,\sigma}^\dagger c_{x,\sigma} \geqslant 0$. Thus, the total Hamiltonian is also positive semi-definite

$$H = H_{\text{hop}} + H_{\text{int}} \geqslant 0. \tag{30}$$

First, we define all-spin-up and all-spin-down states by

$$\Phi_\uparrow = \left(\prod_{x \in \Lambda_o} a_{x,\uparrow}^\dagger \right) \Phi_{\text{vac}} \quad \text{and} \quad \Phi_\downarrow = \left(\prod_{x \in \Lambda_o} a_{x,\downarrow}^\dagger \right) \Phi_{\text{vac}}. \tag{31}$$

We easily verify $H\Phi_\uparrow = 0 = H\Phi_\downarrow$ from the anticommutation relations of $a_{x,\sigma}^\dagger$ and $d_{x,\sigma}$, and therefore these states are ground states of H.

Next, we determine all other ground states. The conditions on a state Φ to be a ground state $H_{\text{hop}}\Phi = 0$ and $H_{\text{int}}\Phi = 0$ are equivalent to

$$d_{x,\sigma}\Phi = 0 \tag{32}$$

for all $x \in \Lambda'$ and

$$c_{y,\uparrow} c_{y,\downarrow} \Phi = 0 \tag{33}$$

for all $y \in \Lambda$ with $\sigma = \uparrow, \downarrow$.

We consider a restricted Hilbert space $\mathcal{H}_M$ spanned by the states with a magnetization M. Since the magnetization is a good quantum number, we find other ground states in each restricted sub-space $\mathcal{H}_M$. Let $\Phi(M)$ be a ground state with a magnetization M. We expand $\Phi(M)$ into the following series:

$$\Phi(M) = \sum_{\mathcal{A},\mathcal{B}} \psi(\mathcal{A},\mathcal{B}) \left(\prod_{x \in \mathcal{A}} a_{x,\uparrow}^\dagger \right) \left(\prod_{y \in \mathcal{B}} a_{y,\downarrow}^\dagger \right) \Phi_{\text{vac}}, \tag{34}$$

where the sum is taken over all subsets $\mathcal{A}, \mathcal{B} \subset \Lambda$ with $|\mathcal{A}| = (L + 2M)/2$ and $|\mathcal{B}| = (L - 2M)/2$, and $\psi(\mathcal{A}, \mathcal{B})$ is a coefficient. The first condition (32) implies that $\psi(\mathcal{A}, \mathcal{B})$ does not vanish only for $\mathcal{A} \cap \Lambda' = \emptyset$ and $\mathcal{B} \cap \Lambda' = \emptyset$. The second condition (33) for $y \in \Lambda_o$ becomes

$$d_{y,\uparrow} d_{y,\downarrow} \Phi(M) = 0, \tag{35}$$

due to the definition of $d_{y,\sigma}$. To satisfy this condition, $\psi(\mathcal{A}, \mathcal{B})$ must vanish for $\mathcal{A} \cap \mathcal{B} \neq \emptyset$. Thus the expansion (34) can be written in the form:

$$\Phi(M) = \sum_{\sigma \in \mathcal{S}_M} \phi(\sigma) \prod_{x \in \Lambda_o} a_{x,\sigma_x}^\dagger \Phi_{\text{vac}} \tag{36}$$

where $\mathcal{S}_M$ is a set of all possible spin configurations $\sigma = (\sigma_x)_{x \in \Lambda_o}$ with a magnetization M:

$$\mathcal{S}_M = \left\{ \sigma = (\sigma_x)_{x \in \Lambda_o} \ \middle| \ \frac{1}{2} \sum_{x \in \Lambda_o} (-1)^{(1-p(\sigma_x)/2)} = M, \ \sigma_x = \uparrow, \downarrow \right\}. \tag{37}$$

In this representation, the second condition (33) for $y \in \Lambda'$ is equivalent to

$$\left(d_{x,\uparrow} d_{x+1,\downarrow} - q d_{x,\downarrow} d_{x+1,\uparrow} \right) \Phi = 0, \tag{38}$$

where we use inverse relations (26) and anticommutation relations (28) with $x = y - (1/2) \in \Lambda_o$. We find that this condition is satisfied if and only if the coefficients satisfy

$$\phi(\sigma) = q^{[p(\sigma_x) - p(\sigma_{x+1})]/2} \phi(\sigma_{x,x+1}) \tag{39}$$

for $x \in \Lambda_o$ with $x < (L-3)/2$, where $\sigma_{x,x+1}$ is a spin configuration obtained by the exchange σ_x and σ_{x+1} in the original configuration σ. This relation implies the uniqueness of the ground state with a fixed magnetization, since two arbitrary spin configurations can be related by successive exchanges of two nearest-neighbour spins. Therefore, the degeneracy of those ground states is exactly the same as that in the SU(2) symmetric model, as in the domain wall ground states in the XXZ model.[1]

If we define S_q^- by

$$S_q^- := \sum_{x \in \Lambda} q^x a_{x,\downarrow}^\dagger d_{x,\uparrow}. \tag{40}$$

The ground state with a magnetization M is obtained by acting this operator $L - 2M$ times on the all-spin-up state

$$\Phi(M) = \left(S_q^-\right)^{L-2M} \Phi_\uparrow. \tag{41}$$

3.2. Another Set of Ground States

To explore the property of the ground state, we write it in a more explicit form as obtained by Gottstein and Werner[15] We define the ground state $\Psi(\zeta)$ by

$$\Psi(\zeta) := \left(\prod_{x \in \Lambda_0} \alpha_x^\dagger(\zeta)\right) \Phi_{\text{vac}} = \sum_{n=0}^{L} \zeta^n (S_q^-)^n \Phi_\uparrow, \tag{42}$$

where $\alpha_x^\dagger(\zeta) := a_{x,\uparrow}^\dagger + \zeta q^x a_{x,\downarrow}^\dagger$. We fix the order of the electron operators depending on their sites. In the XXZ model, the state which corresponds to this ground state $\Psi(\zeta)$ is a pure state in the infinite-volume limit, and therefore we can expect that this state is stable in the infinite-volume limit.[16] Note that the state $\Psi(\zeta)$ defined here in the Hubbard model is not a product state unlike the corresponding state in the XXZ model.

3.3. Spin One-Point Functions

A spin expectation value of a localized spin at site x in the ground state $\Psi(\zeta)$ is written as

$$\langle S_x^{(j)} \rangle_\zeta = \sum_{\sigma,\tau=\uparrow,\downarrow} \frac{(\mathcal{P}^{(j)})_{\sigma,\tau}}{2} \frac{(\Psi(\zeta), c_{x,\sigma}^\dagger c_{x,\tau} \Psi(\zeta))}{\|\Psi(\zeta)\|^2}$$

$$= \sum_{\sigma,\tau=\uparrow,\downarrow} \frac{(\mathcal{P}^{(j)})_{\sigma,\tau}}{2} \frac{(c_{x,\sigma}\Psi(\zeta), c_{x,\tau}\Psi(\zeta))}{\|\Psi(\zeta)\|^2}. \tag{43}$$

The following anticommutation relations with $y \in \Lambda_o$:

$$\{c_{x,\sigma}, \alpha_y^\dagger(\zeta)\} = \lambda\{d_{x,\uparrow}, a_{y,\uparrow}^\dagger + \zeta q^y a_{y,\downarrow}^\dagger\} = \lambda \eta_{x,\sigma}\delta_{x,y}, \tag{44}$$

for $x \in \Lambda_o$, and

$$\{c_{x,\sigma}, \alpha_y^\dagger(\zeta)\} = -\eta_{x,\sigma}\left(q^{-(1/4)}\delta_{x-(1/2),y} + q^{(1/4)}\delta_{x+(1/2),y}\right) \tag{45}$$

for $x \in \Lambda'$ are useful to calculate the expectation value, where $\eta_{x,\uparrow} = 1$ and $\eta_{x,\downarrow} = \zeta q^x$. These anticommutation relations (44) and (45) yield the following equation:

$$c_{x,\sigma}\Psi(\zeta) = \text{sgn}[x]\eta_{x,\sigma}\Psi_x(\zeta), \tag{46}$$

where $\Psi_x(\zeta)$ is defined by

$$\Psi_x(\zeta) = \begin{cases} \lambda\left(\prod_{y \neq x} \alpha_y^\dagger(\zeta)\right)\Phi_{\text{vac}} & \text{if } x \in \Lambda_o, \\ \left(q^{-(1/4)} \prod_{y \neq x-(1/2)} \alpha_y^\dagger(\zeta) - q^{(1/4)} \prod_{y \neq x+(1/2)} \alpha_y^\dagger(\zeta)\right)\Phi_{\text{vac}} & \text{if } x \in \Lambda'. \end{cases} \tag{47}$$

The ordering manner of electron operators determines $\text{sgn}[x] = \pm 1$, which is irrelevant to expectation values of any local operators. Then, the expectation value of $c_{x,\sigma}^\dagger c_{x,\tau}$ for all $x \in \Lambda$ in the ground state $\Psi(\zeta)$ can be written in terms of $\Psi_x(\zeta)$,

$$\langle c_{x,\sigma}^\dagger c_{x,\tau} \rangle_\zeta = \eta_{x,\sigma}^* \eta_{x,\tau} \frac{\|\Psi_x(\zeta)\|^2}{\|\Psi(\zeta)\|^2}. \tag{48}$$

Also a density one-point function $\langle n_x \rangle_\zeta$ can be written in

$$\langle n_x \rangle_\zeta = (1 + |\zeta q^x|^2) \frac{\|\Psi_x(\zeta)\|^2}{\|\Psi(\zeta)\|^2}, \tag{49}$$

where $n_x = n_{x,\uparrow} + n_{x,\downarrow}$. Thus we obtain the representations of spin one-point function at site x in terms of a density one-point function $\langle n_x \rangle_\zeta$ (Fig. 3)

$$\langle S_x^{(1)} \rangle_\zeta = \frac{\langle n_x \rangle_\zeta}{2} \frac{\zeta q^x + (\zeta q^x)^*}{1 + |\zeta q^x|^2},$$

$$\langle S_x^{(2)} \rangle_\zeta = \frac{\langle n_x \rangle_\zeta}{2i} \frac{\zeta q^x - (\zeta q^x)^*}{1 + |\zeta q^x|^2}, \tag{50}$$

$$\langle S_x^{(3)} \rangle_\zeta = \frac{\langle n_x \rangle_\zeta}{2} \frac{1 - |\zeta q^x|^2}{1 + |\zeta q^x|^2}.$$

Here we show briefly the estimates of the density one-point function. We expect the density one-point function in the ground state $\Psi(\zeta)$ takes almost a constant value in Λ_o or in Λ'. Indeed, we will prove this conjecture by evaluating the bounds of this function. Here, we show only the asymptotic forms of the function for large $|x - z|$. The details of evaluating the bounds are given in Appendix A.3. The asymptotic forms of the density one-point function are

$$\lim_{L \to \infty} \langle n_x \rangle_\zeta = \begin{cases} \dfrac{\lambda^2}{\sqrt{\epsilon^2 - 4}}[1 + O(p^{-2|x-z|})] & \text{for} \quad x \in \Lambda_o, \\[3mm] \left(1 - \dfrac{\lambda^2}{\sqrt{\epsilon^2 - 4}}\right)[1 + O(p^{-2|x-z|})] & \text{for} \quad x \in \Lambda', \end{cases} \tag{51}$$

where $\epsilon = \lambda^2 + |q|^{(1/2)} + |q|^{-(1/2)}$ and $p = \min\{r, |q|\}$ for $|q| > 1$ or $p = r$ for $|q| = 1$ with $r = (\epsilon + \sqrt{\epsilon^2 - 4})/2$. In these estimates, the density one-point function is shown to be finite and strictly positive at each site for any finite parameters. Since the density one-point function is nearly equal to the asymptotic value, the behavior of the spin one-point functions is not controlled by the density one-point function besides the oscillation between integer sites and half-odd-integer sites. The spin one-point function $\langle S_x^{(3)} \rangle_\zeta$ vanishes only at $x = -\log_{|q|} |\zeta|$. As discussed for the domain wall ground state in the XXZ models,[2,17,16] the two domains are distinguished by the sign of the local order parameter $\langle S_x^{(3)} \rangle_\zeta$. If the domain wall center is defined by the zeros of $\langle S_x^{(3)} \rangle_\zeta$, then the location of domain wall center
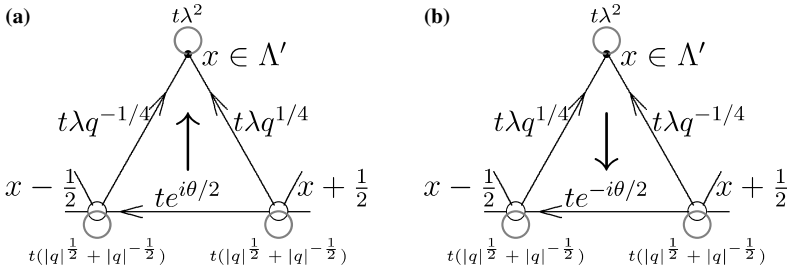
(a)

(b)

Fig. 2.   Arrows and gray circles depict hopping amplitudes and on-site potentials, respectively, for electrons with up (a) and down (b) spins on a unit cell. For complex $q = |q|e^{i\theta}$, the hopping amplitude opposite to the arrow is the complex conjugate of that along the arrow.

is unique. The function $\frac{1}{2}\langle n_x \rangle_\zeta - |\langle S_x^{(3)} \rangle_\zeta|$ decays exponentially as the function of the distance from the center. The domain wall width $1/\log|q|$ is defined by its decay length. In the large λ limit in this model, the density one-point function is exactly equal to 1 for integer sites and 0 for half-odd-integer sites. In this limit and for real $q > 1$, electrons are completely localized on integer sites, and the spin one-point functions are exactly the same as those obtained in the XXZ model defined on Λ_o.

   For a complex $q = |q|e^{i\theta}$ with a real number θ, one can see the spiral structure with a pitch angle θ. The vector $\langle \vec{S}_x \rangle$ is rotated with the angle θx around the third spin axis depending on the site x. Note that this spiral structure of the ground state does not exist in the XXZ model. Though the complex anisotropy parameter $q = e^{i\theta}$ is possible in the XXZ Hamiltonian, the spiral state is no longer a ground state and the corresponding model is described in the Tomonaga–Luttinger liquid without ferromagnetic order. The translational symmetry in the infinite-volume limit is broken by the domain wall or the spiral structure for a finite $\log|\zeta|$. Both symmetries U(1) and $\mathbb{Z}_2$ are broken spontaneously as well. We will discuss the broken symmetries in Section 5.2 (Figs. 2 and 3).

3.4. Electron Two-Point Function

   An electron two-point function $\langle c_{x,\sigma}^\dagger c_{y,\tau} \rangle_\zeta$ can also be written in terms of $\Psi_x(\zeta)$ and $\Psi_y(\zeta)$ by

$$\langle c_{x,\sigma}^\dagger c_{y,\tau} \rangle_\zeta = \frac{(c_{x,\sigma}\Psi(\zeta), c_{y,\tau}\Psi(\zeta))}{\|\Psi(\zeta)\|^2} = (-1)^{y-x}\eta_{x,\sigma}^* \eta_{y,\tau} \frac{(\Psi_x(\zeta), \Psi_y(\zeta))}{\|\Psi(\zeta)\|^2}.$$
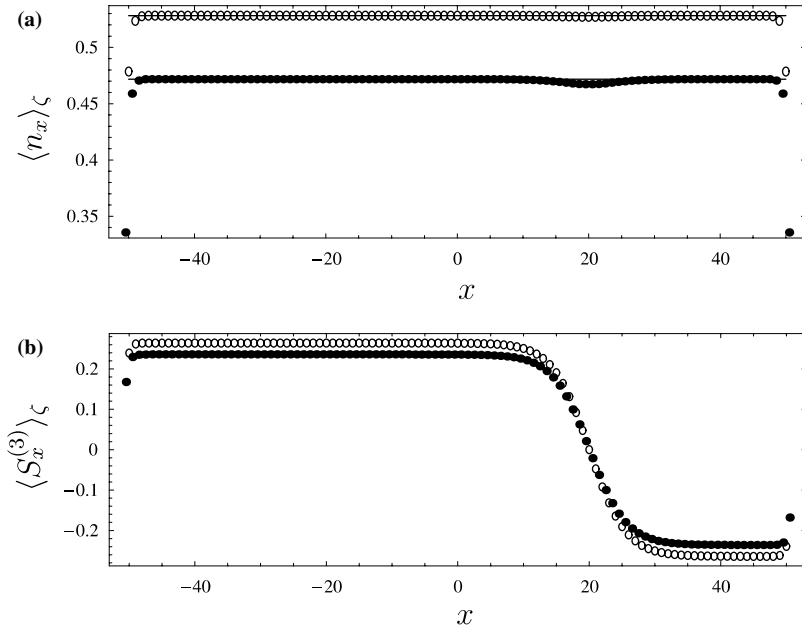
$$(52)$$

Fig. 3. The density one-point function (a) and the spin one-point function (b) are obtained by numerical calculation of rescaled normalization function for $L = 101$, $q = 1.2$, $\zeta = 1.2^{-20}$ and $\lambda = 1.25$. The white circles plot the values on Λ_o and the black dots plot the values on Λ'. The lines in (a) are asymptotic values of density one-point function shown in Eq. (51) for Λ_o and Λ', respectively.

We will see that the electron two-point function can be written in the form

$$\langle c_{x,\sigma}^\dagger c_{y,\tau} \rangle_\zeta = K(x, y, L, \zeta) \frac{\eta_{x,\sigma}^* \eta_{y,\tau}}{\sqrt{1 + |\zeta q^x|^2} \sqrt{1 + |\zeta q^y|^2}} \left(-\frac{e^{i\frac{\theta}{2}}}{r} \right)^{|y-x|}, \quad (53)$$

where $K(x, y, L, \zeta)$ is a slowly varying finite function. This expression shows that the electron two-point function decays exponentially for large $|x - y|$ with the correlation length $1/\log r$.

3.5. Spin Two-Point Functions

The spin two-point functions can be written in the form of

$$\langle S_x^{(j)} S_y^{(k)} \rangle_\zeta = \sum_{\sigma,\tau,\sigma',\tau'} \frac{(\mathcal{P}^{(j)})_{\sigma,\tau}}{2} \frac{(\mathcal{P}^{(k)})_{\sigma',\tau'}}{2} \langle c_{x,\sigma}^\dagger c_{x,\tau} c_{y,\sigma'}^\dagger c_{y,\tau'} \rangle_\zeta. \quad (54)$$

To obtain more explicit form, first, we rewrite the bracket into

$$\langle c_{x,\sigma}^{\dagger} c_{x,\tau} c_{y,\sigma'}^{\dagger} c_{y,\tau'} \rangle_{\zeta} = \frac{(c_{x,\sigma} c_{y,\sigma'} \Psi(\zeta), c_{x,\tau} c_{y,\tau'} \Psi(\zeta))}{\|\Psi(\zeta)\|^2}. \tag{55}$$

Using the anticommutator (44) and (45), we can pick out the spin dependence as coefficients

$$c_{x,\sigma} c_{y,\tau} \Psi(\zeta) = \mathrm{sgn}[x, y] \eta_{x,\sigma} \eta_{y,\tau} \Psi_{x,y}(\zeta), \tag{56}$$

where $\mathrm{sgn}[x, y] = \pm 1$ and

$$
\Psi_{x,y}(\zeta)
$$

$$
= \begin{cases}
\lambda^2 \left[\displaystyle\prod_{w \neq x,y} \alpha_w^{\dagger}(\zeta) \right] \Phi_{\mathrm{vac}} & \text{if } x, y \in \Lambda_{\mathrm{o}}, \\[3em]
\lambda \left[q^{-1/4} \displaystyle\prod_{w \neq x, y-1/2} \alpha_w(\zeta)^{\dagger} - q^{1/4} \displaystyle\prod_{w \neq x, y+1/2} \alpha_w(\zeta)^{\dagger} \right] \Phi_{\mathrm{vac}} & \text{if } x \in \Lambda_{\mathrm{o}}, y \in \Lambda', \\[3em]
\lambda \left[q^{-1/4} \displaystyle\prod_{w \neq x-1/2, y} \alpha_w(\zeta)^{\dagger} - q^{1/4} \displaystyle\prod_{w \neq x+1/2, y} \alpha_w(\zeta)^{\dagger} \right] \Phi_{\mathrm{vac}} & \text{if } y \in \Lambda_{\mathrm{o}}, x \in \Lambda', \\[3em]
\left[q^{-1/2} \displaystyle\prod_{w \neq x-1/2, y-1/2} \alpha_w(\zeta)^{\dagger} + q^{1/2} \displaystyle\prod_{w \neq x+1/2, y+1/2} \alpha_w(\zeta)^{\dagger} \right. \\
\left. \quad - \displaystyle\prod_{w \neq x+1/2, y-1/2} \alpha_w(\zeta)^{\dagger} - \displaystyle\prod_{w \neq x-1/2, y+1/2} \alpha_w(\zeta)^{\dagger} \right] \Phi_{\mathrm{vac}} & \text{if } x, y \in \Lambda'.
\end{cases}
\tag{57}
$$

Then the bracket in (54) is reduced to

$$\langle c_{x,\sigma}^{\dagger} c_{x,\tau} c_{y,\sigma'}^{\dagger} c_{y,\tau'} \rangle_{\zeta} = \eta_{x,\sigma}^* \eta_{x,\tau} \eta_{y,\sigma'}^* \eta_{y,\tau'} \frac{\|\Psi_{x,y}(\zeta)\|^2}{\|\Psi(\zeta)\|^2}. \tag{58}$$

Note that the density two-point function is written

$$\langle n_x n_y \rangle_{\zeta} = (1 + |\zeta q^x|^2)(1 + |\zeta q^y|^2) \frac{\|\Psi_{x,y}(\zeta)\|^2}{\|\Psi(\zeta)\|^2}. \tag{59}$$

We then obtained the representation of the spin two-point function in terms of the density two-point function as well as the spin one-point function written in terms of the density one-point function

$$\langle S_x^{(j)} S_y^{(k)} \rangle_\zeta = \sum_{\sigma,\tau,\sigma',\tau'} \frac{\eta^*_{x,\sigma} \frac{1}{2}(\mathcal{P}^{(j)})_{\sigma,\tau} \eta_{x,\tau}}{1+|\zeta q^x|^2} \frac{\eta^*_{y,\sigma'} \frac{1}{2}(\mathcal{P}^{(k)})_{\sigma',\tau'} \eta_{y,\tau'}}{1+|\zeta q^y|^2} \langle n_x n_y \rangle_\zeta. \qquad (60)$$

If λ is finite, we can rewrite spin two-point function from the spin one-point function (50)

$$\langle S_x^{(j)} S_y^{(k)} \rangle_\zeta = \langle S_x^{(j)} \rangle_\zeta \langle S_y^{(k)} \rangle_\zeta \frac{\langle n_x n_y \rangle_\zeta}{\langle n_x \rangle_\zeta \langle n_y \rangle_\zeta}. \qquad (61)$$

In Section 5, we will find that $\langle n_x \rangle_\zeta$ and $\langle n_x n_y \rangle_\zeta$ exist in the limit $|x| \to \infty$, $|y| \to \infty$ and both. And we will prove

$$\lim_{|x-y| \to \infty} \lim_{L \to \infty} \left(\langle n_x n_y \rangle_\zeta - \langle n_x \rangle_\zeta \langle n_y \rangle_\zeta \right) = 0, \qquad (62)$$

namely,

$$\lim_{|x-y| \to \infty} \lim_{L \to \infty} \frac{\langle n_x n_y \rangle_\zeta}{\langle n_x \rangle_\zeta \langle n_y \rangle_\zeta} = 1. \qquad (63)$$

Thus, we conclude that the spin two-point function satisfies the requirement of the cluster property. We can also find bounds of the density two-point function. The function behaves asymptotically

$$\lim_{L \to \infty} |\langle n_x n_y \rangle_\zeta - \langle n_x \rangle_\zeta \langle n_y \rangle_\zeta| = \begin{cases} O(p^{-2|x-z|}) & \text{if } |y-x| > |x-z| > |y-z|, \\ O(p^{-2|y-z|}) & \text{if } |y-x| > |y-z| > |x-z|, \\ O(p^{-2|y-x|}) & \text{otherwise,} \end{cases} \qquad (64)$$

where p has the same value as that appeared in Eq.(51). We obtain these decay relations in Appendix A.2. The spin two-point function is almost a product of two spin one-point functions. This fact shows that the electron has a good localization property. In other words, $\Psi(\zeta)$ is almost a product state. In the large λ limit, $\langle n_x n_y \rangle_\zeta - \langle n_x \rangle_\zeta \langle n_y \rangle_\zeta$ becomes 0 identically, and then the spin two-point function becomes just a product of two spin one-point functions. This property is the same as domain wall ground state in the XXZ model. We believe that our model is related to the XXZ model if $q \in \mathbb{R}$.

4. NORMALIZATION FUNCTION

In this section, we introduce a normalization function which is useful to evaluate the density one-point function and two-point functions in the ground state $\Psi(\zeta)$.

4.1. Definition of Normalization Function

We define a normalization function $A(x, y; \zeta)$ for $x, y \in \Lambda_o$ with $x \leqslant y$ by

$$A(x, y; \zeta) = \left\| \left(\prod_{z=x}^{y} \alpha_z^{\dagger}(\zeta) \right) \Phi_{\text{vac}} \right\|^2. \tag{65}$$

Note that the norm of the ground state $\Psi(\zeta)$ can be represented by $\|\Psi(\zeta)\|^2 = A(-l, l; \zeta)$, where $l = (L-1)/2$. The anticommutation relation for $\alpha_x(\zeta)$ and $\alpha_y^{\dagger}(\zeta)$ for $x, y \in \Lambda_o$ is

$$\begin{aligned}
\{\alpha_x(\zeta), \alpha_y^{\dagger}(\zeta)\} = \ & (1 + |\zeta q^{x-\frac{1}{2}}|^2) e^{-i\theta/2} \delta_{x-1,y} \\
& + \epsilon(1 + |\zeta q^x|^2) \delta_{x,y} \\
& + (1 + |\zeta q^{x+1/2}|^2) e^{i\theta/2} \delta_{x+1,y},
\end{aligned} \tag{66}$$

where $\epsilon = \lambda^2 + |q|^{1/2} + |q|^{-1/2}$. Using this anticommutation relation, we can obtain the recursion relation for $A(x, y; \zeta)$ in terms of $A(x, y-1; \zeta)$ and $A(x, y-2; \zeta)$, or $A(x+1, y; \zeta)$ and $A(x+2, y; \zeta)$:

$$A(x, y; z) = \epsilon(1 + |\zeta q^y|^2) A(x, y-1; \zeta) - (1 + |\zeta q^{y-1/2}|^2)^2 A(x, y-2; \zeta) \tag{67}$$

or

$$A(x, y; z) = \epsilon(1 + |\zeta q^x|^2) A(x+1, y; \zeta) - (1 + |\zeta q^{x+1/2}|^2)^2 A(x+2, y; \zeta). \tag{68}$$

This forms show that we can obtain $A(x, y; \zeta)$ recursively from the boundary values $A(x, x; \zeta)$ and $A(x, x+1; \zeta)$, or $A(y, y; \zeta)$ and $A(y-1, y; \zeta)$, which are easily calculated as

$$\begin{aligned}
A(x, x; \zeta) \quad &= \epsilon(1 + |\zeta q^x|^2), \\
A(x, x+1; \zeta) &= \epsilon^2(1 + |\zeta q^x|^2)(1 + |\zeta q^{x+1}|^2) - (1 + |\zeta q^{x+1/2}|^2)^2, \\
A(y-1, y; \zeta) &= \epsilon^2(1 + |\zeta q^{y-1}|^2)(1 + |\zeta q^y|^2) - (1 + |\zeta q^{y-1/2}|^2)^2.
\end{aligned} \tag{69}$$

The following formal boundary values are convenient

$$A(x, x-1; \zeta) = A(x+1, x; \zeta) = 1, \qquad A(x, x-2; \zeta) = A(x+2, x; \zeta) = 0.$$
(70)

We can also represent $A(x, y; \zeta)$ in terms of the values $A(w+\delta, y; \zeta)$ and $A(x, w-\delta; \zeta)$ for $w \in \Lambda_o$ and $\delta = 1, 2$ with $x \leqslant w \leqslant y$,

$$\begin{aligned}
A(x, y; \zeta) = {} & -(1+|\zeta q^{w-1/2}|^2)^2 A(x, w-2; \zeta) A(w+1, y; \zeta) \\
& +\epsilon(1+|\zeta q^w|^2) A(x, w-1; \zeta) A(w+1, y; \zeta) \\
& -(1+|\zeta q^{w+1/2}|^2)^2 A(x, w-1; \zeta) A(w+2, y; \zeta),
\end{aligned}$$
(71)

where we can use the formally defined boundary values (70). Note that the relations (67) (68) and (71) are equivalent to each other with the formal boundary values (70). Hereafter, we consider that $A(x, y; \zeta)$ is a function of $x, y \in \mathbb{Z}$ and $\zeta \in \mathbb{C}$ with $y - x \geqslant -2$, which is defined by the recursion relation (67) and formal boundary values (70).

If $|q| = 1$, then the coefficients of the recursion relation (67) are independent of y. This condition gives a great advantage for calculations. We define $\widetilde{A}(x, y)$ by

$$A(x, y; \zeta) = (1+|\zeta|^2)^{y-x+1} \widetilde{A}(x, y).$$
(72)

The recursion relation and boundary values for $\widetilde{A}(x, y)$ are obtained from those for the original function $A(x, y; \zeta)$,

$$\widetilde{A}(x, y) = \epsilon \widetilde{A}(x, y-1) - \widetilde{A}(x, y-2),$$
(73)

and

$$\widetilde{A}(x, x-2) = \widetilde{A}(y+2, y) = 0, \qquad \widetilde{A}(x, x-1) = \widetilde{A}(y+1, y) = 1.$$
(74)

By rewriting this recursion relation to

$$\widetilde{A}(x, y) - r\widetilde{A}(x, y-1) = r^{-1}\left[\widetilde{A}(x, y-1) - r\widetilde{A}(x, y-2)\right],$$
(75)

where $r = (\epsilon + \sqrt{\epsilon^2 - 4})/2$, we can easily obtain the solution

$$\widetilde{A}(x, y) = \frac{r^{y-x+2} - r^{-(y-x+2)}}{r - r^{-1}}.$$
(76)

Then we obtain a compact representation of the function

$$A(x, y; \zeta) = \left[r(1 + |\zeta|^2) \right]^{y-x+1} \frac{1 - r^{-2(y-x+2)}}{1 - r^{-2}}. \tag{77}$$

4.2. Rescaled Normalization Function

In the case of $|q| \neq 1$, we parametrize $|\zeta|$ by a real number z as $|\zeta| = |q|^{-z}$ and define a rescaled normalization function B by

$$A(x, y; \zeta) = \left[\prod_{w=x}^{y} r \left(1 + |q|^{2(w-z)} \right) \right] B(x, y, z). \tag{78}$$

The corresponding recursion relation for the rescaled function obtained from those for the original function (67) is written

$$B(x, y, z) = \frac{1 + r^2}{r^2} B(x, y - 1, z) - \frac{1 - f(y - z - \frac{1}{2})}{r^2} B(x, y - 2, z), \tag{79}$$

where f is defined by

$$f(u) = \frac{(|q|^{(1/2)} - |q|^{-(1/2)})^2}{(|q|^{-u+1/2} + |q|^{u-1/2})(|q|^{-u-1/2} + |q|^{u+1/2})}. \tag{80}$$

This $f(u)$ is an even function of a real number u. At $u = 0$, $f(u)$ takes the maximal value

$$f(0) = \left(\frac{|q|^{1/2} - |q|^{-1/2}}{|q|^{1/2} + |q|^{-1/2}} \right)^2, \tag{81}$$

and for large $|u|$, $f(u)$ is proportional to $|q|^{-2|u|}$ asymptotically. The boundary values are

$$B(x, x - 1, z) = B(x + 1, x, z) = 1, \qquad B(x, x - 2, z) = B(x + 2, x, z) = 0. \tag{82}$$

The relations obtained from (68) and (71) are

$$B(x, y, z) = \frac{1 + r^2}{r^2} B(x + 1, y, z) - \frac{1 - f(x - z + \frac{1}{2})}{r^2} B(x + 2, y, z), \tag{83}$$

and

$$B(x, y, z) = -\frac{1 - f(w - \frac{1}{2} - z)}{r^2} B(x, w - 2, z)B(w + 1, y, z)$$
$$+\frac{1 + r^2}{r^2} B(x, w - 1, z)B(w + 1, y, z)$$
$$-\frac{1 - f(w + \frac{1}{2} - z)}{r^2} B(x, w - 1, z)B(w + 2, y, z), \qquad (84)$$

which are equivalent to the recursion relation (79).

Let us discuss properties of the rescaled function. The function $B(x, y, z)$ is non-negative and monotonically increasing in $y - x$, i.e.,

$$B(x, y, z) \geqslant B(x, y - 1, z), \quad B(x, y, z) \geqslant B(x + 1, y, z). \qquad (85)$$

To obtain these relations, we rewrite the recursion relation (79)

$$B(x, y, z) - B(x, y - 1, z) = \frac{B(x, y - 1, z) - B(x, y - 2, z)}{r^2}$$
$$+\frac{f(y - \frac{1}{2} - z)}{r^2} B(x, y - 2, z). \qquad (86)$$

This representation shows $B(x, y, z) \geqslant B(x, y - 1, z)$ if $B(x, y - 1, z) \geqslant B(x, y - 2, z)$ and $B(x, y - 2, z) \geqslant 0$. From the formal definition (82), we know $B(x, x - 1, z) > B(x, x - 2) = 0$, and therefore, $B(x, y, z) \geqslant B(x, y - 1, z) > 0$. Also $B(x, y, z) \geqslant B(x + 1, y, z)$ is shown almost in the same way. The function $B(x, y, z)$ is invariant under replacing $|q|$ by $|q|^{-1}$ since the coefficients in recursion relation (79) and the formal boundary values are invariant under the replacement of $|q|$ by $|q|^{-1}$.

Though we cannot obtain a compact form of the solution $B(x, y, z)$, we show that $B(x, y, z)$ exists in each limit $y \to \infty$, $x \to -\infty$ or both, respectively, Section 4.3. And we can obtain bounds of the ratio of rescaled normalization functions at two different sites. This bounds are useful to estimate the expectation values in the ground state $\Psi(\zeta)$. We will obtain these bounds in Appendix A.1.

4.3. Convergence of the Rescaled Function

Let us prove that $B(x, y, z)$ converges in each limit $y \to \infty$, $x \to -\infty$ or both. We use the relation (84) for these estimates. The existence of $B(-\infty, \infty, z)$ is shown by this relation, if we show the existence of

$B(x, \infty, z)$ and $B(-\infty, y, z)$. We then concentrate to show the convergence in the limit $y \to \infty$. The convergence in another limit $x \to -\infty$ can be shown in the same way.

We introduce new functions $C(x, y, z)$ and $D(x, y)$ where $x, y \in \mathbb{Z}$ and $z \in \mathbb{R}$ with $y - x \geqslant -2$. These functions are defined by following recursion relations:

$$D(x, y) = \frac{1 + r^2}{r^2} D(x, y - 1) - \frac{1 - g}{r^2} D(x, y - 2), \tag{87}$$

and

$$C(x, y, z) = \frac{1 + r^2}{r^2} C(x, y - 1, z) - \frac{1}{r^2} C(x, y - 2, z)$$
$$+ \frac{f(y - \frac{1}{2} - z)}{r^2} D(x, y - 2) \tag{88}$$

with initial values

$$C(x, x - 2, z) = D(x, x - 2) = 0, \quad C(x, x - 1, z) = D(x, x - 1) = 1, \tag{89}$$

where g is a real parameter. Note that both functions $C(x, y, z)$ and $D(x, y)$ are nonnegative and monotonically increasing for $y - x$ as well as the rescaled normalization function $B(x, y, z)$. Both recursion relations for $C(x, y, z)$ and $D(x, y)$ are much easier than that for $B(x, y, z)$. One can easily obtain a compact representation of the solution $D(x, y)$

$$D(x, y) = \frac{R_+^{y-x+2} - R_-^{y-x+2}}{R_+ - R_-}, \tag{90}$$

where $R_\pm$ are defined by

$$R_+ = 1 + \gamma(g), \qquad R_- = r^{-2} - \gamma(g) \tag{91}$$

with

$$\gamma(g) = \frac{\sqrt{\epsilon^2 - 4 + 4g} - \sqrt{\epsilon^2 - 4}}{2r}. \tag{92}$$

Note that $\gamma(g)$ takes a range $0 < \gamma(g) < r^{-2}$ for $0 < g < 1$.

If g satisfies

$$g \geqslant \max_{x \leqslant w \leqslant y-1} f(w + \tfrac{1}{2} - z), \tag{93}$$

these functions satisfy the following inequalities,

$$B(x, y, z) \leqslant C(x, y, z) \leqslant D(x, y), \tag{94}$$

and

$$B(x, y, z) - B(x, y - 1, z) \leqslant C(x, y, z) - C(x, y - 1, z), \tag{95}$$

for any integers x, y with $x \leqslant y$ and a real number z. To show the second inequality in (94), we calculate $D(x, y) - C(x, y, z)$ using the recursion relation for each

$$
\begin{aligned}
&D(x, y) - C(x, y, z) - [D(x, y-1) - C(x, y-1, z)] \\
&= \frac{1}{r^2}\{D(x, y-1) - C(x, y-1, z) - [D(x, y-2) - C(x, y-2, z)]\} \\
&\quad + \frac{f(y - \tfrac{1}{2} - z)}{r^2}[D(x, y-2) - C(x, y-2, z)] \\
&\quad + \frac{g - f(y - \tfrac{1}{2} - z)}{r^2}D(x, y-2).
\end{aligned} \tag{96}
$$

And note

$$D(x, x) - C(x, x, z) = 0,$$

$$D(x, x+1) - C(x, x+1, z) = \frac{g - f(x + \tfrac{1}{2} - z)}{r^2} > 0. \tag{97}$$

Equations (96) and (97) and the condition (93) imply that $D(x, y) - C(x, y, z)$ is nonnegative and monotonically increasing function of $y - x > 0$ because $D(x, y) > 0$. Thus, we obtain $D(x, y) \geqslant C(x, y, z)$. The remaining two inequalities in (94) and (95) can be shown in the same way.

Now we prove the convergence of $C(x, y, z)$ for $g = f(0)$ in the limit of $y \to \infty$. The convergence of $C(x, y, z)$ and the first inequality in (94) give the convergence of $B(x, y, z)$. We rewrite the recursion relation for $C(x, y, z)$

$$C(x, y, z) - C(x, y-1, z) = \frac{C(x, y-1, z) - C(x, y-2, z)}{r^2}$$

$$+\frac{f(y-\frac{1}{2}-z)}{r^2}\frac{R_+^{y-x}-R_-^{y-x}}{R_+-R_-}. \qquad (98)$$

The final term can be evaluated by

$$\frac{f(y-\frac{1}{2}-z)}{r^2}\frac{R_+^{y-x}-R_-^{y-x}}{R_+-R_-}$$

$$=\left(\frac{R_+}{|q|^2}\right)^y\frac{1}{r^2}\frac{(|q|^{1/2}-|q|^{-1/2})^2}{|q|^{-2z-1}+|q|^{-4y+2z+1}+|q|^{-2y-1}+|q|^{-2y+1}}\frac{R_+^{-x}-R_-^{y-x}/R_+^y}{R_+-R_-}$$

$$<\left(\frac{R_+}{|q|^2}\right)^y\frac{|q|^{2z+1}}{r^2R_+^x}\frac{(|q|^{1/2}-|q|^{-1/2})^2}{R_+-R_-}.$$

From the recursion (98), we have

$$C(x,y,z)-C(x,y-1,z)\leqslant\frac{C(x,y-1,z)-C(x,y-2,z)}{r^2}+\beta^yK_1(x,z), \qquad (99)$$

where $\beta=R_+/|q|^2$ and

$$K_1(x,z)=\frac{|q|^{2z+1}}{r^2R_+^x}\frac{(|q|^{1/2}-|q|^{-1/2})^2}{R_+-R_-}. \qquad (100)$$

Here, we show $\beta<1$ for $|q|>1$ and $g\leqslant f(0)$. We define a function

$$h(\epsilon,g)\equiv\frac{\epsilon+\sqrt{\epsilon^2-4+4g}}{\epsilon+\sqrt{\epsilon^2-4}},$$

which is monotonically decreasing in $\epsilon\geqslant2$ and monotonically increasing in $g\geqslant0$. Then, we represent β as

$$\beta=\frac{1}{|q|^2}h(\epsilon,g). \qquad (101)$$

Since $\epsilon=\lambda^2+|q|^{1/2}+|q|^{-1/2}\geqslant2$, $g\leqslant f(0)$ and $|q|>1$, we obtain

$$\beta=\frac{1}{|q|^2}h(\epsilon,g)\leqslant\frac{1}{|q|^2}h(2,f(0))=\frac{2}{|q|^{3/2}(|q|^{1/2}+|q|^{-1/2})}<1. \qquad (102)$$

Using (99) recursively, we obtain

$$C(x, y, z) - C(x, y-1, z) \leqslant \frac{C(x, y-2, z) - C(x, y-3, z)}{r^4} + \frac{\beta^{y-1}}{r^2} K_1(x, z) + \beta^y K_1(x, z)$$

$$\vdots$$

$$\leqslant \frac{1}{r^{2(y-x+1)}} + \beta^y \frac{(r^2\beta)^{-(y-x+1)} - 1}{(r^2\beta)^{-1} - 1} K_1(x, z)$$

$$= K_2(z-x) r^{-2(y-x)} + K_3(z-x)\beta^{y-x}, \tag{103}$$

where

$$K_2(z-x) = \frac{1}{r^2} \left[1 + \frac{|q|^{2(z-x)+3}}{|q|^2 - r^2 R_+} \frac{(|q|^{1/2} - |q|^{-1/2})^2}{R_+ - R_-} \right], \tag{104}$$

and

$$K_3(z-x) = \frac{R_+ |q|^{2(z-x)+1}}{r^2 R_+ - |q|^2} \frac{(|q|^{1/2} - |q|^{-1/2})^2}{R_+ - R_-}. \tag{105}$$

$K_{2,3}(z-x)$ are finite for fixed x and z. Then we have found that the difference between $C(x, y, z)$ and $C(x, y-1, z)$ decays exponentially as y becomes large if x and z are fixed, and hence $C(x, y, z)$ converges for $y \to \infty$. This fact means that the limit $B(x, \infty, z)$ exists. As discussed before, we easily find the convergence in the limit of $x \to -\infty$ and in the both $y \to \infty$ and $x \to -\infty$.

5. THE INFINITE-VOLUME LIMIT

In this section, we consider the infinite-volume limit. First, we define a ground state ω_ζ in the infinite-volume system on the basis of the ground state $\Psi(\zeta)$ in the finite-volume system. We show that any local operator cannot deform the ground state ω_ζ to a state with lower energy than that of ω_ζ. Next, we remark the symmetries of the model Hamiltonian in the infinite-volume limit and discuss their breakdown. We show also the cluster property of the ground state ω_ζ and excited states above ω_ζ. For the general proof, we use the representations in terms of the rescaled normalization function which is defined only for $|q| \neq 1$ in this paper. For $|q| = 1$, however, the proof is much easier.

5.1. Ground State in the Infinite-Volume System

A state ω_ζ is a linear functional over all local operators for the infinite-volume system defined by

$$\omega_\zeta(O_\mathcal{A}) := \lim_{L\to\infty} \langle O_\mathcal{A}\rangle_\zeta = \lim_{L\to\infty} \frac{(\Psi(\zeta), O_\mathcal{A}\Psi(\zeta))}{\|\Psi(\zeta)\|^2} \tag{106}$$

for an arbitrary local operator $O_\mathcal{A}$ written in electron operators on a finite subset $\mathcal{A}$ of the lattice Λ. We define also other states $\omega_\uparrow$ and $\omega_\downarrow$ for the infinite-volume system corresponding to the finite-volume ground states $\Phi_\uparrow$ and $\Phi_\downarrow$ with the definition (31).

This state ω_ζ is the ground state in the infinite-volume system, if the inequality

$$\lim_{L_1\to\infty} \omega_\zeta(O_\mathcal{A}^\dagger[H_{L_1}, O_\mathcal{A}]) \geqslant 0 \tag{107}$$

is valid for any local operator $O_\mathcal{A}$, where H_{L_1} is a local Hamiltonian defined on a finite chain with length L_1. This inequality implies that any local operator cannot decrease the energy, and hence we can regard the state ω_ζ as a ground state in the infinite-volume limit. In the Hamiltonian (6), there exists $L_2 \in \mathbb{Z}$ for the set $\mathcal{A}$ such that

$$\lim_{L_1\to\infty} [O_\mathcal{A}, H_{L_1}] = [O_\mathcal{A}, H_{L_2}]. \tag{108}$$

Since H_{L_2} is positive semi-definite and $H_{L_2}\Psi(\zeta) = 0$ for any finite L_2,

$$\frac{(\Psi(\zeta), O_\mathcal{A}^\dagger[H_{L_2}, O_\mathcal{A}]\Psi(\zeta))}{\|\Psi(\zeta)\|^2} \geqslant 0 \tag{109}$$

is trivial. Thus, ω_ζ satisfies the inequality (107). We can easily verify that the states $\omega_\uparrow$ and $\omega_\downarrow$ are the ground states of the infinite-volume system.

5.2. Symmetries and Their Breakdown

Here, we consider the U(1), the $\mathbb{Z}_2$ and the translational symmetries of the Hamiltonian in the infinite-volume system. These symmetries are defined for a finite system by Eqs.(15) (21) and (22). Let $\mathcal{G}$ be one symmetry group of them. An arbitrary transformation $X \in \mathcal{G}$ can act on an

arbitrary local operator O_A written in electron operators on a finite subset A of the lattice Λ

$$O_A \longmapsto X(O_A).$$

The Hamiltonian of the infinite-volume system has a symmetry $\mathcal{G}$, if a local Hamiltonian H_{L_1} defined on a finite chain with a length L_1 has the following transformation property

$$\lim_{L_1 \to \infty} X([O_A, H_{L_1}]) = \lim_{L_1 \to \infty} [X(O_A), H_{L_1}] \tag{110}$$

for any $X \in \mathcal{G}$ and for any local operator O_A. Thus, our model has U(1), $\mathbb{Z}_2$ and translational symmetries defined by the mappings (15) (21) and (22) in the infinite-volume limit.

The spin one-point functions in the infinite-volume limit are shown

$$\omega_\zeta\left(S_x^{(1)}\right) = \frac{\omega_\zeta(n_x)}{2} \frac{q^{x-z} + (q^{x-z})^*}{1 + |q^{x-z}|^2},$$

$$\omega_\zeta\left(S_x^{(2)}\right) = \frac{\omega_\zeta(n_x)}{2i} \frac{q^{x-z} - (q^{x-z})^*}{1 + |q^{x-z}|^2}, \tag{111}$$

$$\omega_\zeta\left(S_x^{(3)}\right) = \frac{\omega_\zeta(n_x)}{2} \frac{1 - |q^{x-z}|^2}{1 + |q^{x-z}|^2}.$$

All the symmetries of the model are spontaneously broken in the ground state ω_ζ, since

$$\omega_\zeta(U_\varphi(S_x^{(1)})) \neq \omega_\zeta(S_x^{(1)}), \qquad \omega_\zeta(\Pi(S_x^{(3)})) \neq \omega_\zeta(S_x^{(3)}), \qquad \omega_\zeta(T_u(S_x^{(3)})) \neq \omega_\zeta(S_x^{(3)}). \tag{112}$$

On the other hand, the spin one-point functions in the all-spin-up and all-spin-down ground states are

$$\omega_\uparrow(S_x^{(1)}) = 0 = \omega_\uparrow(S_x^{(2)}), \qquad \omega_\uparrow(S_x^{(3)}) = \tfrac{1}{2}\omega_\uparrow(n_x), \tag{113}$$

$$\omega_\downarrow(S_x^{(1)}) = 0 = \omega_\downarrow(S_x^{(2)}), \qquad \omega_\downarrow(S_x^{(3)}) = -\tfrac{1}{2}\omega_\downarrow(n_x). \tag{114}$$

Therefore, the ground states $\omega_\uparrow$ and $\omega_\downarrow$ have the $U(1)$ and the translational symmetries. The broken symmetry is only $\mathbb{Z}_2$ in $\omega_\uparrow$ and $\omega_\downarrow$.

5.3. The Cluster Property

First, we give a precise definition of the cluster property. Let $\mathcal{A}$ and $\mathcal{B}$ be finite subsets of Λ. And we denote a translation of $\mathcal{A}$ by $\mathcal{A}+x$ for $x \in \Lambda$:

$$\mathcal{A}+x := \{x+y \mid y \in \mathcal{A}\}. \tag{115}$$

We say that a state ω has the cluster property, if the truncated correlation function of arbitrary observables $O_\mathcal{A}$ and $O_\mathcal{B}$ on arbitrary finite subsets $\mathcal{A}, \mathcal{B} \subset \Lambda$ satisfies

$$\lim_{x \to \infty} [\omega(O_\mathcal{A} O_{\mathcal{B} \pm x}) - \omega(O_\mathcal{A})\omega(O_{\mathcal{B} \pm x})] = 0 \tag{116}$$

and

$$\lim_{x \to \infty} [\omega(O_{\mathcal{A} \pm x} O_\mathcal{B}) - \omega(O_{\mathcal{A} \pm x})\omega(O_\mathcal{B})] = 0. \tag{117}$$

Here we prove the cluster property of the ground state ω_ζ. Let $O_{\mathcal{A},\mathcal{B}}$ is a basis of local operators

$$O_{\mathcal{A},\mathcal{B}}(\sigma_\mathcal{A}, \sigma_\mathcal{B}) = \left(\prod_{x \in \mathcal{A}} d^\dagger_{x,\sigma_x} \right) \left(\prod_{y \in \mathcal{B}} d_{y,\sigma_y} \right), \tag{118}$$

where $\sigma_\mathcal{A} = (\sigma_x)_{x \in \mathcal{A}}$ with $\sigma_x = \uparrow, \downarrow$. The relation given in (26) means that any local operator can be written in the form

$$\sum_j \sum_{\sigma_{\mathcal{A}_j}, \sigma_{\mathcal{B}_j}} \phi_j(\sigma_{\mathcal{A}_j}, \sigma_{\mathcal{B}_j}) O_{\mathcal{A}_j, \mathcal{B}_j}(\sigma_{\mathcal{A}_j}, \sigma_{\mathcal{B}_j}), \tag{119}$$

where $\phi_j(\sigma_{\mathcal{A}_j}, \sigma_{\mathcal{B}_j}) \in \mathbb{C}$ is a coefficient. Hereafter we do not write the spin dependence explicitly for simplicity. Note that the summation in this representation (119) consists of finite terms. The cluster property is shown if we find

$$\lim_{x \to \infty} \left[\omega_\zeta(O_{\mathcal{A},\mathcal{B}} O_{\mathcal{C}+x, \mathcal{D}+x}) - \omega_\zeta(O_{\mathcal{A},\mathcal{B}})\omega_\zeta(O_{\mathcal{C}+x, \mathcal{D}+x}) \right] = 0 \tag{120}$$

and

$$\lim_{x \to \infty} \left[\omega_\zeta(O_{\mathcal{A}-x, \mathcal{B}-x} O_{\mathcal{C},\mathcal{D}}) - \omega_\zeta(O_{\mathcal{A}-x, \mathcal{B}-x})\omega_\zeta(O_{\mathcal{C},\mathcal{D}}) \right] = 0 \tag{121}$$

for any finite subsets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ of Λ. The following two lemmas are useful to prove these equalities: (120) and (121).

Lemma 1.   Let $x_j \in \mathcal{A}$ $(j=1, 2, \ldots, |\mathcal{A}|)$ and $y_k \in \mathcal{B}$ $(k=1, 2, \ldots, |\mathcal{B}|)$ be elements labeled with the monotonically increasing property $x_j < x_{j+1}$ and $y_k < y_{k+1}$. If $\langle O_{\mathcal{A},\mathcal{B}}(\sigma_\mathcal{A}, \sigma_\mathcal{B}) \rangle_\zeta$ does not vanish, then $\mathcal{A}$ and $\mathcal{B}$ satisfy the following conditions: $\mathcal{A}, \mathcal{B} \subset \Lambda_\mathrm{o}$, $|\mathcal{A}| = |\mathcal{B}|$, $x_j < y_j \Rightarrow y_j < x_{j+1}$ and $y_j < x_j \Rightarrow x_j < y_{j+1}$ for any $j = 1, \ldots, |\mathcal{A}| - 1$. In this case, $\langle O_{\mathcal{A},\mathcal{B}} \rangle_\zeta$ can be written in the form of

$$\langle O_{\mathcal{A},\mathcal{B}}(\sigma_\mathcal{A}, \sigma_\mathcal{B}) \rangle_\zeta = \frac{B(-l, u-1, z) B(v+1, l, z)}{B(-l, l, z)} T_{\mathcal{A},\mathcal{B}}(\sigma_\mathcal{A}, \sigma_\mathcal{B}), \quad (122)$$

where $u = \min\{x_1, y_1\}$ and $v = \max\{x_{|\mathcal{A}|}, y_{|\mathcal{B}|}\}$. $T_{\mathcal{A},\mathcal{B}}$ and its translation $T_{\mathcal{A}\pm w, \mathcal{B} \pm w}$ are finite even in the limit of $|w| \to \infty$.

Lemma 2.   We consider four finite subsets $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ of Λ_o, and define lattice sites $x = \min \mathcal{A} \cup \mathcal{B}$, $y = \max \mathcal{A} \cup \mathcal{B}$, $u = \min \mathcal{C} \cup \mathcal{D}$ and $v = \max \mathcal{C} \cup \mathcal{D}$ with $y < u$. If $\langle O_{\mathcal{A},\mathcal{B}} O_{\mathcal{C},\mathcal{D}} \rangle_\zeta$ does not vanish, we have to consider only the following two cases. If either $\langle O_{\mathcal{A},\mathcal{B}} \rangle_\zeta$ or $\langle O_{\mathcal{C},\mathcal{D}} \rangle_\zeta$ does not vanish, then it can be written in the form:

$$\langle O_{\mathcal{A},\mathcal{B}} O_{\mathcal{C},\mathcal{D}} \rangle_\zeta = \frac{B(-l, x-1, z) B(y+1, u-1, z) B(v+1, l, z)}{B(-l, l, z)} T_{\mathcal{A},\mathcal{B}} T_{\mathcal{C},\mathcal{D}}. \quad (123)$$

The finite coefficient $T_{\mathcal{A},\mathcal{B}}$ is identical to $T_{\mathcal{A},\mathcal{B}}(\sigma_\mathcal{A}, \sigma_\mathcal{B})$ in Lemma 1. We have abbreviated the spin dependence. If both $\langle O_{\mathcal{A},\mathcal{B}} \rangle_\zeta$ and $\langle O_{\mathcal{C},\mathcal{D}} \rangle_\zeta$ vanish, the following function decays exponentially

$$\langle O_{\mathcal{A},\mathcal{B}} O_{\mathcal{C},\mathcal{D}} \rangle_\zeta = \frac{B(-l, x-1, z) B(v+1, l, z)}{B(-l, l, z)} \left(-\frac{e^{i\frac{\theta}{2}}}{r} \right)^{u-y} \tilde{T}_{\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}}. \quad (124)$$

$\tilde{T}_{\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}}$ and its translation $\tilde{T}_{\mathcal{A}-w',\mathcal{B}-w',\mathcal{C}+w,\mathcal{D}+w}$ is finite even in the limit $|w| \to \infty$, $|w'| \to \infty$ and both.

We can easily prove these lemmas in practical calculations of expectation values in the ground state $\Psi(\zeta)$ as for the density two-point function or the electron two-point function. They will appear in Appendices A.4 and A.5.

Proof of the Cluster Property. Now we prove the cluster property of the ground state ω_ζ and excited states above ω_ζ. In the case of $\langle O_{\mathcal{A},\mathcal{B}} \rangle_\zeta = 0$, Lemma 2 implies $\langle O_{\mathcal{A},\mathcal{B}} O_{\mathcal{C},\mathcal{D}} \rangle_\zeta \to 0$ in the limit of $(\max \mathcal{A} \cup \mathcal{B} - \min \mathcal{C} \cup \mathcal{D}) \to \infty$. Next, we consider the case of $\langle O_{\mathcal{A},\mathcal{B}} \rangle_\zeta \neq 0$ and $\langle O_{\mathcal{C},\mathcal{D}} \rangle_\zeta \neq 0$. The lemmas 1 and 2 ensure that the truncated correlation function can be written in the form of:

$$
\begin{aligned}
\langle O_{\mathcal{A},\mathcal{B}} & O_{\mathcal{C},\mathcal{D}} \rangle_\zeta - \langle O_{\mathcal{A},\mathcal{B}} \rangle_\zeta \langle O_{\mathcal{C},\mathcal{D}} \rangle_\zeta \\
&= T_{\mathcal{A},\mathcal{B}} T_{\mathcal{C},\mathcal{D}} \frac{B(-l,x-1,z)B(v+1,l,z)}{B(-l,l,z)^2} [B(y+1,u-1,z) \\
&\quad \times B(-l,l,z) - B(-l,u-1,z)B(y+1,l,z)].
\end{aligned}
$$

$$(125)$$

We already know that $T_{\mathcal{C}+w,\mathcal{D}+w}$ and $\lim_{l\to\infty} B(v+w+1,l,z)/B(-l,l,z)$ are finite even for $w \to \infty$ and

$$
\lim_{u\to\infty} \lim_{l\to\infty} \left(B(y+1,u-1,z)B(-l,l,z) - B(-l,u-1,z)B(y+1,l,z) \right) = 0.
$$

This relation enables us to show the requirements (120) and (121). Thus, the cluster property of the ground state ω_ζ has been proved. We can prove the cluster property of an arbitrary excited state above the ground state ω_ζ basically in the same approach, since any excited state can be obtained by acting some local operators written in a finite summation of the basis (118) on the ground state ω_ζ. In Appendix A.2, we will find the following bound for a sufficiently large u:

$$
\left| \lim_{l\to\infty} \left(B(y+1,u-1,z)B(-l,l,z) - B(-l,u-1,z)B(y+1,l,z) \right) \right| < Cp^{-2(u-w)}
$$

$$(126)$$

with a positive number C independent of u for $w = \max\{y,z\}$, $y = \max \mathcal{A} \cup \mathcal{B}$ and $u = \min \mathcal{C} \cup \mathcal{D}$. Here, we define $p = \min\{|q|,r\}$ for $|q| > 1$, or $p = r$ for $|q| = 1$. This bound and the representation (124) mean that all truncated correlation functions decay exponentially. And their decay lengths are less than $(2 \log p)^{-1}$.

The proof of the cluster property or other evaluations of expectation values for the ground states $\omega_\uparrow$ and $\omega_\downarrow$ are much easier than those for the domain wall ground state ω_ζ because the recursion relations of the normalization functions for $\Phi_\uparrow$ and $\Phi_\downarrow$ are exactly solvable.

APPENDIX A

Here, we show the estimates of the rescaled normalization function to evaluate the density one-point function and several correlation functions. In the case of $|q|=1$, explicit and compact representations of expectation values are easily obtained by the exact formula of normalization function (77). Also in the case of $|q| \neq 1$, the dominant part of the two-point functions is the same as those for $|q|=1$. Here, we show the evaluation only for $|q| > 1$ without loss of generality.

A.1. Estimates of the Rescaled Normalization Function

Here, we show useful inequalities

$$1 - G_+(x, y, z) \leqslant \frac{B(x, y, z)}{B(x, y+1, z)} \leqslant 1, \qquad 1 - G_-(x, y, z) \leqslant \frac{B(x, y, z)}{B(x-1, y, z)} \leqslant 1 \tag{A.1}$$

for $x < y$. The positivity $1 - G_\pm(x, y, z) > 0$ is guaranteed by the definition

$$G_\pm(x, y, z) = \min \left\{ \frac{1}{r^2+1}, \tilde{G}_\pm(x, y, z) \right\}, \tag{A.2}$$

where the functions $\tilde{G}_\pm(x, y, z)$ are defined by

$$\tilde{G}_+(x, y, z) = \frac{r^{-2(y-x+1)}}{1+r^2} + (|q|^{1/2} - |q|^{-1/2})^2$$

$$\times \begin{cases} |q|^{-2(x-z-\frac{1}{2})} \dfrac{r^{-2(y-x+1)} - |q|^{-2(y-x+1)}}{|q|^2 - r^2} & \text{if } x \geqslant z - \frac{1}{2}, \\[2ex] |q|^{-2(z-y-3/2)} \dfrac{1 - |q|^{-2(y-x+1)} r^{-2(y-x+1)}}{|q|^2 r^2 - 1} & \text{if } y \leqslant z - \frac{1}{2}, \\[2ex] -r^{-2(y-x+1)} \dfrac{|q|^{-2(z-x-1/2)}}{r^2 |q|^2 - 1} + \dfrac{|q|^{-2(y-z+1/2)}}{r^2 - |q|^2} & \\[2ex] + |q| r^{-2(y-z'+1)} \left(\dfrac{|q|^{2(z'-z)}}{r^2 |q|^2 - 1} + \dfrac{|q|^{2(z-z')}}{|q|^2 - r^2} \right) & \text{otherwise,} \end{cases} \tag{A.3}$$

and

$$\tilde{G}_-(x, y, z) = \frac{r^{-2(y-x+1)}}{1+r^2} + (|q|^{1/2} - |q|^{-1/2})^2$$

$$\times \begin{cases} |q|^{-2(x-z-3/2)}\dfrac{1-|q|^{-2(y-x+1)}r^{-2(y-x+1)}}{|q|^2r^2-1} & \text{if } x \geqslant z+\tfrac{1}{2}, \\[3mm] |q|^{-2(z-y-1/2)}\dfrac{r^{-2(y-x+1)}-|q|^{-2(y-x+1)}}{|q|^2-r^2} & \text{if } y \leqslant z+\tfrac{1}{2}, \\[3mm] -r^{-2(y-x+1)}\dfrac{|q|^{-2(y-z-1/2)}}{r^2|q|^2-1}+\dfrac{|q|^{-2(z-x+1/2)}}{r^2-|q|^2} \\[3mm] +|q|r^{-2(z'-x+1)}\left(\dfrac{|q|^{2(z-z')}}{r^2|q|^2-1}+\dfrac{|q|^{2(z'-z)}}{|q|^2-r^2}\right) & \text{otherwise,} \end{cases} \tag{A.4}$$

where z' is an integer defined by $z' \in \mathbb{Z} \cap [z-\tfrac{1}{2}, z+\tfrac{1}{2})$. These inequalities give the estimates of the expectation values. Note that in a limit $\lambda \to \infty$, r goes to ∞, and therefore, $B(x, y-1, z)/B(x, y, z) \to 1$ for arbitrary x, y and z.

Now we prove the inequalities (A.1). The upper bounds are obvious, because the function $B(x, y, z)$ is monotonically increasing for $y-x$. The remaining task is to obtain the lower bounds.

Here, we treat only $B(x, y, z)/B(x, y+1, z)$. The lower bound for $B(x, y, z)/B(x-1, y, z)$ can be obtained in almost the same approach. First, we show $B(x, y, z)/B(x, y+1, z) \geqslant 1-(r^2+1)^{-1}$. Using the relation (79), we obtain

$$\frac{B(x, y, z)}{B(x, y+1, z)} = \left[\frac{1+r^2}{r^2} - \frac{1-f(y+\tfrac{1}{2}-z)}{r^2}\frac{B(x, y-1, z)}{B(x, y, z)}\right]^{-1}. \tag{A.5}$$

Note $B(x, y-1, z)/B(x, y, z) > 0$ and $1-f(y+\tfrac{1}{2}-z) > 0$. Then we obtain

$$\frac{B(x, y, z)}{B(x, y+1, z)} > \frac{r^2}{1+r^2} = 1 - \frac{1}{r^2+1}. \tag{A.6}$$

To obtain another lower bound, we rewrite the recursion relation

$$B(x, y+1, z) - B(x, y, z) = \frac{1}{r^2}\left[B(x, y, z) - B(x, y-1, z)\right]$$
$$+ \frac{f(y+\tfrac{1}{2}-z)}{r^2}B(x, y-1, z). \tag{A.7}$$

And using this relation recursively, we obtain

$$B(x, y+1, z) - B(x, y, z) = \frac{1}{r^4}\left[B(x, y-1, z) - B(x, y-2, z)\right]$$

$$+\frac{f(y-\frac{1}{2}-z)}{r^4}B(x,y-2,z)$$

$$+\frac{f(y+\frac{1}{2}-z)}{r^2}B(x,y-1,z)$$

$$\vdots$$

$$=\frac{1}{r^{2(y-x+2)}}+\sum_{j=0}^{y-x}\frac{f(y+\frac{1}{2}-j-z)}{r^{2(j+1)}}$$

$$\times B(x,y-j-1,z)$$

$$(A.8)$$

by using $B(x,x-2,z)=0$. Then we obtain

$$\frac{B(x,y,z)}{B(x,y+1,z)}=1-\frac{1}{r^{2(y-x+2)}}\frac{1}{B(x,y+1,z)}$$

$$-\sum_{j=0}^{y-x}\frac{f(y+\frac{1}{2}-j-z)}{r^{2(j+1)}}\frac{B(x,y-j-1,z)}{B(x,y+1,z)}$$

$$\geqslant 1-\frac{r^{-2(y-x+1)}}{1+r^2}-\sum_{j=0}^{y-x}\frac{f(y+\frac{1}{2}-j-z)}{r^{2(j+1)}} \qquad (A.9)$$

by using

$$B(x,y+1,z)\geqslant\frac{1+r^2}{r^2}, \qquad \frac{B(x,y-j-1,z)}{B(x,y+1,z)}\leqslant 1, \qquad (A.10)$$

for $y\geqslant x$. Note the inequality

$$f(u)\leqslant(|q|^{1/2}-|q|^{-1/2})^2|q|^{-2|u|}. \qquad (A.11)$$

We can evaluate the summation in the final line of (A.9). There are three cases. First case is $x\geqslant z-\frac{1}{2}$. Since $y+\frac{1}{2}-j-z>0$ for $0\leqslant j\leqslant y-x$, we obtain

$$\sum_{j=0}^{y-x}\frac{f(y+\frac{1}{2}-j-z)}{r^{2(j+1)}}\leqslant\frac{(|q|^{1/2}-|q|^{-1/2})^2}{r^2}\sum_{j=0}^{y-x}\frac{|q|^{-2(y-j-z+1/2)}}{r^{2j}}$$

$$=(|q|^{1/2}-|q|^{-1/2})^2|q|^{-2(x-z-1/2)}$$

$$\times \frac{r^{-2(y-x+1)} - |q|^{-2(y-x+1)}}{|q|^2 - r^2}. \tag{A.12}$$

For the second case $y \leqslant z - \frac{1}{2}$, we obtain

$$\sum_{j=0}^{y-x} \frac{f(y + \frac{1}{2} - j - z)}{r^{2(j+1)}} \leqslant \frac{(|q|^{1/2} - |q|^{-1/2})^2}{r^2} \sum_{j=0}^{y-x} \frac{|q|^{2(y-j-z+1/2)}}{r^{2j}}$$

$$= (|q|^{1/2} - |q|^{-1/2})^2 |q|^{-2(z-y-3/2)}$$

$$\times \frac{1 - |q|^{-2(y-x+1)} r^{-2(y-x+1)}}{|q|^2 r^2 - 1}, \tag{A.13}$$

because $y + \frac{1}{2} - j - z < 0$ for $0 \leqslant j \leqslant y - x$. For the third case $x < z - \frac{1}{2} < y$, we introduce an integer n such that $0 \leqslant y - n - z + \frac{1}{2}$ and $y - (n+1) - z + \frac{1}{2} < 0$. Then the summation can be evaluated by

$$\sum_{j=0}^{y-x} \frac{f(y + \frac{1}{2} - j - z)}{r^{2(j+1)}} \leqslant \frac{(|q|^{1/2} - |q|^{-1/2})^2}{r^2} \left[\sum_{j=0}^{n} \frac{|q|^{-2(y-j-z+1/2)}}{r^{2j}} \right.$$

$$+ \sum_{j=n+1}^{y-x} \frac{|q|^{2(y-j-z+1/2)}}{r^{2j}} \right]$$

$$= (|q|^{1/2} - |q|^{-1/2})^2 \left[-r^{-2(y-x+1)} \frac{|q|^{-2(z-x-1/2)}}{r^2 |q|^2 - 1} \right.$$

$$+ \frac{|q|^{-2(y-z+1/2)}}{r^2 - |q|^2}$$

$$+ |q| r^{-2(y-z'+1)} \left(\frac{|q|^{2(z'-z)}}{r^2 |q|^2 - 1} + \frac{|q|^{2(z-z')}}{|q|^2 - r^2} \right) \right], \tag{A.14}$$

where $z' := y - n \in \mathbb{Z} \cap [z - \frac{1}{2}, z + \frac{1}{2})$. Then we obtain another lower bound

$$\frac{B(x, y, z)}{B(x, y+1, z)} \geqslant 1 - \frac{r^{-2(y-x+1)}}{1 + r^2} - (|q|^{1/2} - |q|^{-1/2})^2$$

$$\times \begin{cases} |q|^{-2(x-z-1/2)} \dfrac{r^{-2(y-x+1)}-|q|^{-2(y-x+1)}}{|q|^2-r^2} & \text{if } x \geqslant z - \tfrac{1}{2}, \\[2ex] |q|^{-2(z-y-3/2)} \dfrac{1-|q|^{-2(y-x+1)}r^{-2(y-x+1)}}{|q|^2r^2-1} & \text{if } y \leqslant z - \tfrac{1}{2}, \\[2ex] -r^{-2(y-x+1)} \dfrac{|q|^{-2(z-x-1/2)}}{r^2|q|^2-1} + \dfrac{|q|^{-2(y-z+1/2)}}{r^2-|q|^2} \\[2ex] +|q|r^{-2(y-z'+1)}\left(\dfrac{|q|^{2(z'-z)}}{r^2|q|^2-1} + \dfrac{|q|^{2(z-z')}}{|q|^2-r^2}\right) & \text{otherwise.} \end{cases}$$

$$(A.15)$$

A.2. Estimates of the Truncated Correlation Functions

Here, we prove the inequality (126). That inequality means that all truncated correlation functions decay exponentially. First, we show the simple form of bounds of derivation of two rescaled normalization function. For any x and z, there is a positive number C_0 independent of y such that for sufficient large y

$$\frac{B(x, y, z)}{B(x, y+1, z)} < 1 - C_0 p^{-2(y+1-w)} \qquad (A.16)$$

with $w = \max\{z, x\}$. Here, we define $p = \min\{|q|, r\}$ for $|q| > 1$, or $p = r$ for $|q| = 1$. We can obtain this bound easily from the bounds (A.1). And following bound is trivial

$$\frac{B(x, y+1, z)}{B(x, y, z)} < 1 + C_1 p^{-2(y+1-w)}. \qquad (A.17)$$

Here, the finite positive number C_1 is also independent of y.

Let us prove the inequality (126).

$$B(-l, l, z)B(x+1, y-1, z) - B(-l, y-1, l, z)B(x+1, l, z)$$
$$= B(-l, l, z)B(x+1, y-1, z)\left[1 - \frac{B(-l, y-1, l, z)B(x+1, l, z)}{B(-l, l, z)B(x+1, y-1, z)}\right]. \qquad (A.18)$$

We concentrate to obtain bounds of

$$\frac{B(-l, y-1, l, z)B(x+1, l, z)}{B(-l, l, z)B(x+1, y-1, z)}, \qquad (A.19)$$

since we have already known that the pre-factor $B(-l, l, z)B(x+1, y-1, z)$ is finite even in the limit of $l \to \infty$ and $y \to \infty$. First we estimate an

upper bound. Note that $B(x, y, z)$ is a monotonically increasing function of $y - x$. We can evaluate

$$\frac{B(-l, y-1, l, z)}{B(-l, l, z)} \frac{B(x+1, l, z)}{B(x+1, y-1, z)} < \frac{B(x+1, l, z)}{B(x+1, y-1, z)}. \tag{A.20}$$

And using Eq. (A.17), we obtain

$$\frac{B(x+1, l, z)}{B(x+1, y-1, z)} = \frac{B(x+1, l, z)}{B(x+1, l-1, z)} \frac{B(x+1, l-1, z)}{B(x+1, l-2, z)} \cdots \frac{B(x+1, y, z)}{B(x+1, y-1, z)}$$

$$< \prod_{v=y}^{l} (1 + C_1 p^{-2(v-w)}). \tag{A.21}$$

To estimate this, we take the logarithm. Then we can evaluate

$$\log\left[\prod_{v=y}^{l} (1 + C_1 p^{-2(v-w)})\right] = \sum_{v=y}^{l} \log(1 + C_1 p^{-2(v-w)})$$

$$< C_2 \sum_{v=y}^{l} p^{-2(v-w)} = C_3(p^{-2(y-w)} - p^{-2(l-w)}) \tag{A.22}$$

for some positive numbers C_2 and C_3 independent of y. Here, we have used the mean-value theorem to obtain the second line. Then we obtain an upper bound

$$\frac{B(-l, y-1, l, z)B(x+1, l, z)}{B(-l, l, z)B(x+1, y-1, z)} < e^{C_3(p^{-2(y-w)} - p^{-2(l-w)})}$$

$$< 1 + C_4(p^{-2(y-w)} - p^{-2(l-w)}) \tag{A.23}$$

for a positive number C_4. We also can evaluate a lower bound

$$\frac{B(-l, y-1, l, z)B(x+1, l, z)}{B(-l, l, z)B(x+1, y-1, z)} > 1 - C_5(p^{-2(y-z)} - p^{-2(l-z)}) \tag{A.24}$$

with a positive number C_5 independent of y. Then we have obtained the bound for a sufficiently large y

$$\left|\lim_{l\to\infty}\left[B(-l, l, z)B(x+1, y-1, z) - B(-l, y-1, l, z)B(x+1, l, z)\right]\right| < Cp^{-2(y-w)}. \tag{A.25}$$

A.3. Estimates of the Density One-Point Function

Now we can estimate the density one-point function. We first estimate the factor

$$\frac{B(-l, x-1, z)B(x+1, l, z)}{B(-l, l, z)}, \tag{A.26}$$

which appears in the representations of the density one-point function both for integer sites (A.30) and half-odd-integer sites (A.31). Using the relation (84), we obtain

$$\frac{B(-l, x-1, z)B(x+1, l, z)}{B(-l, l, z)} = \left[\frac{1+r^2}{r^2} - \frac{1-f(x-\frac{1}{2}-z)}{r^2}\frac{B(-l, x-2, z)}{B(-l, x-1, z)}\right.$$
$$\left. - \frac{1-f(x+\frac{1}{2}-z)}{r^2}\frac{B(x+2, l, z)}{B(x+1, l, z)}\right]^{-1}.$$

This form and the inequality (A.1) enable us to estimate the upper bound

$$\frac{B(-l, x-1, z)B(x+1, l, z)}{B(-l, l, z)} < \frac{r^2}{r^2-1} = \frac{r}{\sqrt{\epsilon^2-4}},$$

by using

$$\frac{B(-l, x-2, z)}{B(-l, x-1, z)} \leqslant 1, \qquad \frac{B(x+2, l, z)}{B(x+1, l, z)} \leqslant 1$$

and

$$1 - f(x \pm \tfrac{1}{2} - z) < 1.$$

For the lower bound, we obtain the inequality

$$\frac{B(-l, x-1, z)B(x+1, l, z)}{B(-l, l, z)} > \left\{\frac{1+r^2}{r^2} - \frac{1-f(x-\frac{1}{2}-z)}{r^2}[1 - G_+(-l, x-2, z)]\right.$$
$$\left. - \frac{1-f(x+\frac{1}{2}-z)}{r^2}[1 - G_-(x+2, l, z)]\right\}^{-1}$$
$$= \frac{r}{\sqrt{\epsilon^2-4}}\frac{1}{1+G_1(x, l, z)} \tag{A.27}$$

by using

$$\frac{B(-l, x-2, z)}{B(-l, x-1, z)} \geqslant 1 - G_+(-l, x-2, z), \quad \frac{B(x+2, l, z)}{B(x+1, l, z)} \geqslant 1 - G_-(x+2, l, z)$$

$$(A.28)$$

with

$$
\begin{aligned}
G_1(x, l, z) = \ & \frac{1}{r^2-1}\bigg\{ f(x-\tfrac{1}{2}-z)[1-G_+(-l, x-2, z)] \\
& + f(x+\tfrac{1}{2}-z)[1-G_-(x+2, l, z)] + G_+(-l, x-2, z) \\
& + G_-(x+2, l, z)\bigg\}.
\end{aligned}
$$

$$(A.29)$$

The representation of the density one-point function in terms of the rescaled normalization function is

$$\langle n_x \rangle_\zeta = \frac{\lambda^2}{r}\frac{B(-l, x-1, z)B(x+1, l, z)}{B(-l, l, z)}.$$

$$(A.30)$$

for $x \in \Lambda_o$ or

$$
\begin{aligned}
\langle n_x \rangle_\zeta = \frac{1}{r}\bigg[& \frac{|q|^{x-z}+|q|^{-x+z}}{|q|^{x-z-1/2}+|q|^{-x+z+1/2}}\frac{B(-l, x-\tfrac{3}{2}, z)B(x+\tfrac{1}{2}, l, z)}{B(-l, l, z)} \\
& \times \bigg\{ 1 - \frac{1}{r}\frac{|q|^{x-z}+|q|^{-x+z}}{|q|^{x-z+1/2}+|q|^{-x+z-1/2}}\frac{B(x+\tfrac{3}{2}, l, z)}{B(x+\tfrac{1}{2}, l, z)} \bigg\} \\
& + \frac{|q|^{x-z}+|q|^{-x+z}}{|q|^{x-z+1/2}+|q|^{-x+z-1/2}}\frac{B(-l, x-\tfrac{1}{2}, z)B(x+\tfrac{3}{2}, l, z)}{B(-l, l, z)} \\
& \times \bigg\{ 1 - \frac{1}{r}\frac{|q|^{x-z}+|q|^{-x+z}}{|q|^{x-z-1/2}+|q|^{-x+z+1/2}}\frac{B(-l, x-\tfrac{2}{3}, z)}{B(-l, x-\tfrac{1}{2}, z)} \bigg\} \bigg].
\end{aligned}
$$

$$(A.31)$$

for $x \in \Lambda'$.

From the representation (A.30), we find the bounds

$$\frac{\lambda^2}{\sqrt{\epsilon^2-4}}\frac{1}{1+G_1(x, l, z)} < \langle n_x \rangle_\zeta < \frac{\lambda^2}{\sqrt{\epsilon^2-4}}.$$

$$(A.32)$$

$G_1(x, l, z)$ consists of terms which decay exponentially as $l - |x|$, $l - |z|$ or $|x - z|$ become large. And for sufficiently large $|x - z|$, $\lim_{l \to \infty} G_1(x, l, z)$ is proportional to $p^{-2|x-z|}$ asymptotically

$$\lim_{l \to \infty} G_1(x, l, z) = O(p^{-2|x-z|}), \tag{A.33}$$

where $p = \min\{|q|, r\}$.

Next, we estimate the density one-point function for a site in Λ'. For upper bound of the density one-point function, we evaluate

$$
\begin{aligned}
\langle n_x \rangle_\zeta \; < \; & \frac{1}{\sqrt{\epsilon^2 - 4}} \left[\frac{|q|^{x-z} + |q|^{-x+z}}{|q|^{x-z-1/2} + |q|^{-x+z+1/2}} \right. \\
& \times \left\{ 1 - \frac{1}{r} \frac{|q|^{x-z} + |q|^{-x+z}}{|q|^{x-z+1/2} + |q|^{-x+z-1/2}} \left[1 - G_-(x + \tfrac{1}{2}, l, z) \right] \right\} \\
& + \frac{|q|^{x-z} + |q|^{-x+z}}{|q|^{x-z+1/2} + |q|^{-x+z-1/2}} \\
& \times \left. \left\{ 1 - \frac{1}{r} \frac{|q|^{x-z} + |q|^{-x+z}}{|q|^{x-z-1/2} + |q|^{-x+z+1/2}} \left[1 - G_+(-l, x - \tfrac{3}{2}, z) \right] \right\} \right] \\[6pt]
= \; & \frac{1}{\sqrt{\epsilon^2 - 4}} \\
& \times \left[\frac{|q|^{x-z} + |q|^{-x+z}}{|q|^{x-z-1/2} + |q|^{-x+z+1/2}} \frac{r|q|^{x-z+1/2} + r|q|^{-x+z-1/2} - |q|^{x-z} - |q|^{-x+z}}{r(|q|^{x-z+1/2} + |q|^{-x+z-1/2})} \right. \\
& + \frac{|q|^{x-z} + |q|^{-x+z}}{|q|^{x-z+1/2} + |q|^{-x+z-1/2}} \frac{r|q|^{x-z-1/2} + r|q|^{-x+z+1/2} - |q|^{x-z} - |q|^{-x+z}}{r(|q|^{x-z-1/2} + |q|^{-x+z+1/2})} \\
& + \frac{1}{r} \frac{|q|^{x-z} + |q|^{-x+z}}{|q|^{x-z+1/2} + |q|^{-x+z-1/2}} G_-(x + \tfrac{1}{2}, l, z) \\
& + \left. \frac{1}{r} \frac{|q|^{x-z} + |q|^{-x+z}}{|q|^{x-z-1/2} + |q|^{-x+z+1/2}} G_+(-l, x - \tfrac{3}{2}, z) \right] \\[6pt]
= \; & \frac{1}{\sqrt{\epsilon^2 - 4}} \left[\frac{|q|^{x-z} + |q|^{-x+z}}{(|q|^{x-z-1/2} + |q|^{-x+z+1/2})(|q|^{x-z+1/2} + |q|^{-x+z-1/2})} \right. \\
& \times \left. \left\{ |q|^{x-z}(|q|^{1/2} + |q|^{-1/2} - 2r^{-1}) + |q|^{-x+z}(|q|^{1/2} + |q|^{-1/2} - 2r^{-1}) \right\} \right. \\
& + \left. G_2^{+'}(x, l, z) \right] \\[6pt]
= \; & \frac{1}{\sqrt{\epsilon^2 - 4}} \left[\frac{(\sqrt{\epsilon^2 - 4} - \lambda^2)(|q|^{x-z} + |q|^{-x+z})^2}{(|q|^{x-z-1/2} + |q|^{-x+z+1/2})(|q|^{x-z+1/2} + |q|^{-x+z-1/2})} \right\} \\
& + \left. G_2^{+'}(x, l, z) \right] \\[6pt]
= \; & \left(1 - \frac{\lambda^2}{\sqrt{\epsilon^2 - 4}} \right) [1 + G_2^+(x, l, z)], \tag{A.34}
\end{aligned}
$$

where we have used $|q|^{(1/2)} + |q|^{-(1/2)} = \epsilon - \lambda^2$, and we determine a new function G_2^+, which includes the terms which are proportional to $|q|^{-|x-z|}$. And we obtain the lower bound

$$
\langle n_x \rangle_\zeta > \frac{1}{\sqrt{\epsilon^2 - 4}} \Bigg[\frac{|q|^{x-z} + |q|^{-x+z}}{|q|^{x-z-1/2} + |q|^{-x+z+1/2}} \left\{ 1 - \frac{1}{r} \frac{|q|^{x-z} + |q|^{-x+z}}{|q|^{x-z+1/2} + |q|^{-x+z-1/2}} \right\}
$$

$$
\times \frac{1}{1 + G_1(x - \tfrac{1}{2}, l, z)}
$$

$$
+ \frac{|q|^{x-z} + |q|^{-x+z}}{|q|^{x-z+1/2} + |q|^{-x+z-1/2}} \left\{ 1 - \frac{1}{r} \frac{|q|^{x-z} + |q|^{-x+z}}{|q|^{x-z-1/2} + |q|^{-x+z+1/2}} \right\}
$$

$$
\times \frac{1}{1 + G_1(x + \tfrac{1}{2}, l, z)} \Bigg]
$$

$$
= \frac{1}{\sqrt{\epsilon^2 - 4}} \frac{(|q|^{x-z} + |q|^{-x+z})(|q|^{1/2} + |q|^{-1/2} - 2r^{-1})}{(|q|^{x-z-1/2} + |q|^{-x+z+1/2})(|q|^{x-z+1/2} + |q|^{-x+z-1/2})}
$$

$$
\times \left\{ |q|^{x-z} \frac{1}{1 + G_1(x - \tfrac{1}{2}, l, z)} + |q|^{-x+z} \frac{1}{1 + G_1(x + \tfrac{1}{2}, l, z)} \right\} \tag{A.35}
$$

$$
= \left(1 - \frac{\lambda^2}{\sqrt{\epsilon^2 - 4}} \right) [1 - G_2^-(x, l, z)]. \tag{A.36}
$$

Hence we obtain bounds of the density one-point function

$$
\left(1 - \frac{\lambda^2}{\sqrt{\epsilon^2 - 4}} \right) [1 - G_2^-(x, l, z)] < \langle n_x \rangle_\zeta < \left(1 - \frac{\lambda^2}{\sqrt{\epsilon^2 - 4}} \right) [1 + G_2^+(x, l, z)]. \tag{A.37}
$$

These $G_2^\pm(x, l, z)$ consist of terms which decay exponentially for $l - |x|$, $l - |z|$ or $|x - z|$. In the infinite-volume limit, these functions are asymptotically equal to

$$
\lim_{l \to \infty} G_2^\pm(x, l, z) = O(p^{-2|x-z|}) \tag{A.38}
$$

for large $|x - z|$. Inequality (A.35) shows $G_2^- < 1$, and hence the density one-point function does not vanish for all $x \in \Lambda$.

A.4. Estimates of the Electron Two-Point Function

We can represent the electron two-point functions in the following form

$$
\langle c_{x,\sigma}^\dagger c_{y,\tau} \rangle_\zeta = K'(x, y, l, z) \frac{\eta_{x,\sigma}^* \eta_{y,\tau}}{\sqrt{1 + |q|^{2(x-z)}} \sqrt{1 + |q|^{2(y-z)}}} \left(-\frac{e^{i\frac{\theta}{2}}}{r} \right)^{y-x}, \tag{A.39}
$$

where

$$
\begin{aligned}
K'(x, y, l, z) &= \frac{\lambda^2}{r} \sqrt{\frac{1 + |q|^{2(y-z)}}{1 + |q|^{2(x-z)}}} \left(\prod_{w=x+1}^{y} \frac{1 + |q|^{2(w-z-1/2)}}{1 + |q|^{2(w-z)}} \right) \\
&\quad \times \frac{B(-l, x-1, z) B(y+1, l, z)}{B(-l, l, z)}
\end{aligned}
\tag{A.40}
$$

for $x, y \in \Lambda_0$. We can easily obtain $K'(x, y, l, z)$ for other $x, y \in \Lambda$, but they are complicated forms. Since the rescaled normalization function converges in the limit $l \to \infty$, $K'(x, y, l, z)$ exists in the infinite-volume limit. Apparently, $K'(x, y, l, z)$ seems to diverge according to the factor $\sqrt{1 + |q|^{2(y-z)}}$. But we can evaluate

$$
\begin{aligned}
\prod_{w=v+1}^{y} \frac{1 + |q|^{2(w-z-1/2)}}{1 + |q|^{2(w-z)}} &= |q|^{-(y-v)} \prod_{w=v+1}^{y} \frac{|q|^{-2(w-z)+1} + 1}{|q|^{-2(w-z)} + 1} \\
&\leqslant |q|^{-(y-v)} \prod_{w=v+1}^{y} \frac{|q|^{-2(w-z)+2} + 1}{|q|^{-2(w-z)} + 1} \\
&= |q|^{-(y-v)} \frac{1 + |q|^{-2(v-z)}}{1 + |q|^{-2(y-z)}},
\end{aligned}
\tag{A.41}
$$

and then the factor

$$
\begin{aligned}
&\sqrt{\frac{1 + |q|^{2(y-z)}}{1 + |q|^{2(x-z)}}} \left(\prod_{w=x+1}^{y} \frac{1 + |q|^{2(w-z-1/2)}}{1 + |q|^{2(w-z)}} \right) \\
&< \sqrt{\frac{|q|^{-2(y-v)} + |q|^{2(v-z)}}{1 + |q|^{2(x-z)}}} \left(\prod_{w=x+1}^{v} \frac{1 + |q|^{2(w-z-1/2)}}{1 + |q|^{2(w-z)}} \right) \frac{1 + |q|^{-2(v-z)}}{1 + |q|^{-2(y-v)}}
\end{aligned}
\tag{A.42}
$$

is finite for any x and y. Thus, we have found that $K'(x, y, l, z)$ is finite for $x, y \in \Lambda_0$ even in the limit of $l \to \infty$. We can show that $K'(x, y, l, z)$ is finite also for other $x, y \in \Lambda$. The expression (A.39) shows that the electron two-point function decays exponentially as $y - x$ becomes large with a decay length $1/\log r$ because of the finiteness of $K'(x, y, l, z)$. Thus, the electron two-point function $\omega_\zeta(c_{x,\sigma}^\dagger c_{y,\tau})$ satisfies the requirement of the cluster property defined in Section 5.5.

A.5. Results of the Density Two-Point Function

We can estimate the density two-point function as well as the one-point function. The density two-point function is written in

$$\langle n_x n_y \rangle_\zeta = \frac{\lambda^4}{r^2} \frac{B(-l, x-1, z) B(x+1, y-1, z) B(y+1, l, z)}{B(-l, l, z)}. \quad \text{(A.43)}$$

for $x, y \in \Lambda_{\mathrm{o}}$. We can easily obtain representation for other $x, y \in \Lambda$, but it has a complicated form. The representations in terms of the rescaled normalization function give us good estimates. From the relation (84), we obtain

$$
\begin{aligned}
B(-l, l, z) =\ & \left(1 + \frac{1}{r^2}\right)^2 B(-l, x-1, z) B(x+1, y-1, z) B(y+1, l, z) \\
& - \left(1 + \frac{1}{r^2}\right) \Bigg[\frac{1 - f(x - \frac{1}{2} - z)}{r^2} B(-l, x-2, z) B(x+1, y-1, z) \\
& \times B(y+1, l, z) \\
& + \frac{1 - f(x + \frac{1}{2} - z)}{r^2} B(-l, x-1, z) B(x+2, y-1, z) B(y+1, l, z) \\
& + \frac{1 - f(y - \frac{1}{2} - z)}{r^2} B(-l, x-1, z) B(x+1, y-2, z) B(y+1, l, z) \\
& + \frac{1 - f(y + \frac{1}{2} - z)}{r^2} B(-l, x-1, z) B(x+1, y-1, z) B(y+2, l, z) \Bigg] \\
& + \frac{1 - f(x - \frac{1}{2} - z)}{r^2} \frac{1 - f(y - \frac{1}{2} - z)}{r^2} B(-l, x-2, z) B(x+1, y-2, z) B(y+1, l, z) \\
& + \frac{1 - f(x - \frac{1}{2} - z)}{r^2} \frac{1 - f(y + \frac{1}{2} - z)}{r^2} B(-l, x-2, z) B(x+1, y-1, z) B(y+2, l, z) \\
& + \frac{1 - f(x + \frac{1}{2} - z)}{r^2} \frac{1 - f(y - \frac{1}{2} - z)}{r^2} B(-l, x-1, z) B(x+2, y-2, z) B(y+1, l, z) \\
& + \frac{1 - f(x + \frac{1}{2} - z)}{r^2} \frac{1 - f(y + \frac{1}{2} - z)}{r^2} B(-l, x-1, z) B(x+2, y-1, z) B(y+2, l, z).
\end{aligned}
$$
$$\text{(A.44)}$$

From this relation, we evaluate the density two-point function

$$
\lim_{l \to \infty} \langle n_x n_y \rangle_\zeta =
\begin{cases}
\dfrac{\lambda^4}{\epsilon^2 - 4} G_3(x, y, z) & \text{if } x, y \in \Lambda_{\mathrm{o}}, \\[2ex]
\left(1 - \dfrac{\lambda^2}{\sqrt{\epsilon^2 - 4}}\right)^2 G_4(x, y, z) & \text{if } x, y \in \Lambda', \\[2ex]
\dfrac{\lambda^2}{\sqrt{\epsilon^2 - 4}} \left(1 - \dfrac{\lambda^2}{\sqrt{\epsilon^2 - 4}}\right) G_5(x, y, z) & \text{otherwise.}
\end{cases}
\quad \text{(A.45)}
$$

Functions G_3, G_4 and G_5 are definite functions which approach to 1 exponentially as $|y - x|$, $|y - z|$ and $|x - z|$ become large.

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